

# On the Rate of Convergence of Weak Euler Approximation for Non-degenerate SDEs

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## Abstract

The paper estimates the rate of convergence of the weak Euler approximation for solutions to SDEs driven by point and martingale measures, with Hölder continuous coefficients. The equation considered has a non-degenerate main part whose jump intensity measure is absolutely continuous with respect to the Lévy measure of a spherically-symmetric stable process. It includes the nondegenerate diffusions and SDEs driven by Lévy processes.

*Keywords:* Lévy processes, stochastic differential equations, weak Euler approximation

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## 1. Introduction

In this paper we consider the weak Euler approximation for solutions to SDEs driven by point and martingale measures. It is a continuation of [15] where some Markov Itô processes were approximated. Let  $\alpha \in (0, 2]$  be fixed. In a complete probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  with a filtration  $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$  of  $\sigma$ -algebras satisfying the usual conditions, we consider an  $\mathbb{F}$ -adapted  $d$ -dimensional stochastic process  $X_t, t \in [0, T]$ , solving for  $t \in [0, T]$

$$\begin{aligned} X_t = & X_0 + \int_0^t a_\alpha(X_{s-}) ds + \int_0^t b_\alpha(X_{s-}) dW_s \\ & + \int_0^t \int_{|y| > 1} c(X_{s-}) h_\alpha(X_{s-}, \frac{y}{|y|}) y p_0(ds, dy) \\ & + \int_0^t \int_{|y| \leq 1} c(X_{s-}) h_\alpha(X_{s-}, \frac{y}{|y|}) y q_0(ds, dy) \\ & + \int_0^t \int_{U_1^c} l_\alpha(X_{s-}, v) p(ds, dv) + \int_0^t \int_{U_1} l_\alpha(X_{s-}, v) q(ds, dv), \end{aligned} \tag{1}$$

where  $W_t, t \in [0, T]$ , is a  $d$ -dimensional  $\mathbb{F}$ -adapted standard Wiener process,  $p_0(dt, dy)$  and  $p(dt, dv)$  are independent Poisson point measures on  $[0, T] \times \mathbf{R}_0^d$  ( $\mathbf{R}_0^d = \mathbf{R}^d \setminus \{0\}$ ) and  $[0, T] \times U$  respectively with

$$\begin{aligned} q_0(dt, dy) &= p_0(dt, dy) - \frac{dy}{|y|^{d+\alpha}} dt, \\ q(dt, dv) &= p(dt, dv) - \pi(dv) dt, \end{aligned}$$

being the corresponding martingale measures and

$$\begin{aligned} a_\alpha(x) &= \mathbf{1}_{\{\alpha \in (0,1)\}} \left( \int_{|y| \leq 1} c(x) h_\alpha(x, \frac{y}{|y|}) y \frac{dy}{|y|^{d+\alpha}} + \int_{U_1} l_\alpha(x, v) \pi(dv) \right) \\ &\quad + \mathbf{1}_{\{\alpha=1\}} \left( a(x) + \int_{U_1} l_\alpha(x, v) \pi(dv) \right) \\ &\quad + \mathbf{1}_{\{\alpha \in (1,2)\}} \left( a(x) - \int_{|y| > 1} c(x) h_\alpha(x, \frac{y}{|y|}) y \frac{dy}{|y|^{d+\alpha}} \right), \\ b_\alpha(x) &= \mathbf{1}_{\{\alpha=2\}} b(x), \end{aligned}$$

The coefficient functions  $a = (a^i)_{1 \leq i \leq d}, \alpha \in [1, 2], c = (c^{ij})_{1 \leq i, j \leq d}, h_\alpha, \alpha \in (0, 2)$ , and  $b = (b^{ij})_{1 \leq i, j \leq d}, \alpha = 2$ , are measurable and bounded,  $\pi(dv)$  is a non-negative  $\sigma$ -finite measure on a measurable space  $(U, \mathcal{U})$ : there is a sequence  $U_n \in \mathcal{U}$  such that  $U = \bigcup_n U_n^c$  and  $\pi(U_n^c) < \infty$  for each  $n$ . We assume that  $l_\alpha, \alpha \in (0, 2]$  is measurable and  $\int_{U_1} |l_\alpha(x, v)|^\alpha \pi(d, v)$  is bounded. A class of strong Markov processes satisfying (1) is constructed, for example, in [13], [1] (see references therein as well). In particular, (1) covers a large class of SDEs driven by Lévy processes (see subsection 2.3 below).

The process defined in (1) is used as a mathematical model for random dynamic phenomena in applications from fields such as finance and insurance, to capture continuous and discontinuous uncertainty. It naturally arises in stochastic differential equations driven by Lévy processes as well (see subsection 2.3 below). For many applications, the practical computation of functionals of the type  $F = \mathbf{E}g(X_T)$  and  $F = \mathbf{E} \int_0^T f(X_s) ds$  plays an important role. For instance in finance, derivative prices can be expressed by such functionals. One possibility to numerically approximate  $F$  is given by the discrete time Monte-Carlo simulation of the Itô process  $X$ . The simplest discrete time approximation of  $X$  that can be used for such Monte-Carlo methods is the weak Euler approximation.

Let the time discretization  $\{\tau_i, i = 0, \dots, n_T\}$  of the interval  $[0, T]$  with maximum step size  $\delta \in (0, 1)$  be a partition of  $[0, T]$  such that  $0 = \tau_0 < \tau_1 < \dots < \tau_{n_T} = T$  and  $\max_i(\tau_i - \tau_{i-1}) \leq \delta$ . The Euler approximation of

$X$  is an  $\mathbb{F}$ -adapted stochastic process  $Y = \{Y_t\}_{t \in [0, T]}$  defined for  $t \in [0, T]$  by the stochastic equation

$$\begin{aligned}
Y_t &= X_0 + \int_0^t \int_{\mathbf{R}_0^d} c(Y_{\tau_{i_s}}) h_\alpha(Y_{\tau_{i_s}}, \frac{y}{|y|}) y p_0(ds, dy) \\
&\quad + \int_0^t \int l_\alpha(Y_{\tau_{i_s}}, v) p(ds, dv) \text{ if } \alpha \in (0, 1), \\
Y_t &= X_0 + \int_0^t a(Y_{\tau_{i_s}}) ds + \int_0^t \int_{|y| > 1} c(Y_{\tau_{i_s}}) h_\alpha(Y_{\tau_{i_s}}, \frac{y}{|y|}) y p_0(ds, dy) \\
&\quad + \int_0^t \int_{|y| \leq 1} c(Y_{\tau_{i_s}}) h_\alpha(Y_{\tau_{i_s}}, \frac{y}{|y|}) y q_0(ds, dy) + \int_0^t \int l_\alpha(Y_{\tau_{i_s}}, v) p(ds, dv) \text{ if } \alpha = 1, \\
Y_t &= X_0 + \int_0^t a(Y_{\tau_{i_s}}) ds + \int_0^t \int_{\mathbf{R}_0^d} c(Y_{\tau_{i_s}}) h_\alpha(Y_{\tau_{i_s}}, \frac{y}{|y|}) y q_0(ds, dy) \\
&\quad + \int_0^t \int_{U_1} l_\alpha(Y_{\tau_{i_s}}, v) q(ds, dv) + \int_0^t \int_{U_1^c} l_\alpha(Y_{\tau_{i_s}}, v) p(ds, dv) \text{ if } \alpha \in (1, 2), \\
Y_t &= X_0 + \int_0^t a(Y_{\tau_{i_s}}) ds + \int_0^t b(Y_{\tau_{i_s}}) dW_s \\
&\quad + \int_0^t \int_{U_1} l_2(Y_{\tau_{i_s}}, v) q(ds, dv) + \int_0^t \int_{U_1^c} l_2(Y_{\tau_{i_s}}, v) p(ds, dv) \text{ if } \alpha = 2,
\end{aligned} \tag{2}$$

where  $\tau_{i_s} = \tau_i$  if  $s \in [\tau_i, \tau_{i+1})$ ,  $i = 0, \dots, n_T - 1$ . Contrary to those in (1), the coefficients in (2) are piecewise constants in each time interval of  $[\tau_i, \tau_{i+1})$ .

The weak Euler approximation  $Y$  is said to converge with order  $\kappa > 0$  if for each bounded smooth function  $g$  with bounded derivatives, there exists a constant  $C$ , depending only on  $g$ , such that

$$|\mathbf{E}g(Y_T) - \mathbf{E}g(X_T)| \leq C\delta^\kappa,$$

where  $\delta > 0$  is the maximum step size of the time discretization.

The cases in which the coefficients are smooth, especially for diffusion processes ( $\alpha = 2, \pi = 0$ ), have been considered by many authors. Milstein (see [16, 17]) was one of the first to study the order of weak convergence for diffusion processes (8) with  $\alpha = 2$  and derived  $\kappa = 1$ . Talay in [21, 22] investigated a class of the second order approximations for diffusion processes. For Itô processes with jump components, Mikulevičius & Platen showed the first-order convergence in the case in which the coefficient functions possess fourth-order continuous derivatives (see [10]). In Platen and Kloeden & Platen (see [6, 18]), not only Euler but also higher order approximations

were studied as well. Protter and Talay in [20] considered the weak Euler approximation for

$$X_t = X_0 + \int_0^t C(X_{s-}) dZ_s, t \in [0, T], \quad (3)$$

where  $Z_t = (Z_t^1, \dots, Z_t^m)$  is a Lévy process and  $C = (C^{ij})_{1 \leq i \leq d, 1 \leq j \leq m}$  is a measurable and bounded function. They showed the order of convergence  $\kappa = 1$ , provided that  $c$  and  $g$  are smooth and the Lévy measure of  $Z$  has finite moments of sufficiently high order. Because of that, the main theorems in [20] do not apply to (8). On the other hand, (1) with non-degenerate  $c(x), x \in \mathbf{R}^d$ , do not cover (3) which can degenerate completely.

In general, the coefficients and the test function  $g$  do not always have the smoothness properties assumed in the papers cited above. Mikulevičius & Platen (see [11]) proved that there is still some order of convergence of the weak Euler approximation for non-degenerate diffusion processes ((8) with  $\alpha = 2$ ) under Hölder conditions on the coefficients and  $g$ . In Kubilius & Platen [9], Platen & Bruti-Liberati [19] a weak Euler approximation was considered in the case of a non-degenerate diffusion processes with a finite number of jumps in finite time intervals.

This paper is a follow-up to [15], where  $X_t$  was a Markov Itô process solving a martingale problem. In this paper, we derive the rate of convergence for (1) under  $\beta$ -Hölder conditions on the coefficients. As in [15] (see [21] as well), we use the solution to the backward Kolmogorov equation associated with  $X_t$  and the one-step estimates derived in [15].

In the following Section 2, we introduce assumptions and state the main result. In Section 3, we present the essential technical results. The main theorem is proved in Section 4.

## 2. Notation and Main Result

### 2.1. Notation

Denote  $H = [0, T] \times \mathbf{R}^d$ ,  $\mathbf{N} = \{0, 1, 2, \dots\}$ ,  $\mathbf{R}_0^d = \mathbf{R}^d \setminus \{0\}$ . For  $x, y \in \mathbf{R}^d$ , write  $(x, y) = \sum_{i=1}^d x_i y_i$ ,  $|x| = \sqrt{(x, x)}$  and  $|B| = \sum_{i=1}^d |B^{ii}|$ ,  $B \in \mathbf{R}^{d \times d}$ .

Let  $S^{d-1}$  denote the unit sphere in  $\mathbf{R}^d$ , with  $\mu_{d-1}$  being the Lebesgue measure on it.

$C_b^\infty(H)$  is the set of all functions  $u$  on  $H$  such that for all  $t \in [0, T]$  the function  $u(t, x)$  is infinitely differentiable in  $x$  and for every multiindex  $\gamma \in \mathbf{N}^d$ ,

$$\sup_{(t,x) \in H} |\partial_x^\gamma u(t, x)| < \infty,$$

where

$$\partial_x^\gamma u(t, x) = \frac{\partial^{|\gamma|}}{\partial^{\gamma_1 x_1} \dots \partial^{\gamma_d x_d}} u(t, x).$$

$C_0^\infty(G)$  is the set of all infinitely differentiable functions on an open set  $G \subseteq \mathbf{R}^d$  with compact support.  $\mathcal{S}(\mathbf{R}^d)$  is the Schwartz space of rapidly decaying smooth functions.

Denote

$$\begin{aligned} \partial_t u(t, x) &= \frac{\partial}{\partial t} u(t, x), \\ \partial_i u(t, x) &= \frac{\partial}{\partial x_i} u(t, x), i = 1, \dots, d, \\ \partial_{ij}^2 u(t, x) &= \frac{\partial^2}{\partial x_i \partial x_j} u(t, x), i, j = 1, \dots, d, \\ \partial_x u(t, x) &= \nabla u(t, x) = (\partial_1 u(t, x), \dots, \partial_d u(t, x)), \\ \partial^k u(t, x) &= (\partial^\gamma u(t, x))_{|\gamma|=k}, k \in \mathbf{N}. \end{aligned}$$

For  $\alpha \in (0, 2)$ , write

$$\begin{aligned} |\partial|^\alpha v(x) &= -\mathcal{F}^{-1}[|\xi|^\alpha \mathcal{F}v(\xi)](x), \\ |\partial|^2 v(x) &= \Delta v(x) = \sum_{i=1}^d \partial_{ii}^2 v(x). \end{aligned}$$

where  $\mathcal{F}$  denotes the Fourier transform with respect to  $x \in \mathbf{R}^d$  and  $\mathcal{F}^{-1}$  is the inverse Fourier transform, i.e.,

$$\mathcal{F}v(\xi) = \int_{\mathbf{R}^d} e^{-i(\xi, x)} u(x) dx, \quad \mathcal{F}^{-1}v(x) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{i(\xi, x)} v(\xi) d\xi.$$

$C = C(\cdot, \dots, \cdot)$  denotes constants depending only on quantities appearing in parentheses. In a given context the same letter is (generally) used to denote different constants depending on the same set of arguments.

## 2.2. Assumptions and Main Result

Assume  $m_\alpha(x, y) = |h_\alpha(x, y)|^\alpha, x, y \in \mathbf{R}^d, \alpha \in (0, 2)$ , and its partial derivatives  $\partial_y^\gamma m_\alpha(x, y), |\gamma| \leq d_0 = [\frac{d}{2}] + 1$  are continuous in  $(x, y)$ . Moreover,  $m_\alpha(x, y)$  is homogeneous in  $y$  with index zero, and  $m_1(x, y)$  is symmetric in  $y$ :  $m_1(x, -y) = m_1(x, y), x \in \mathbf{R}^d, y \in S^{d-1}$ .

For  $\beta = [\beta]^- + \{\beta\}^+ > 0$ , where  $[\beta]^- \in \mathbf{N}$  and  $\{\beta\}^+ \in (0, 1]$ , let  $C^\beta(H)$  denote the space of measurable functions  $u$  on  $H$  such that the norm

$$\begin{aligned} |u|_\beta &= \sum_{|\gamma| \leq [\beta]^-} \sup_{(t,x) \in H} |\partial_x^\gamma u(t,x)| + \sup_{|\gamma| = [\beta]^-, t, x \neq \tilde{x}} \frac{|\partial_x^\gamma u(t,x) - \partial_x^\gamma u(t, \tilde{x})|}{|x - \tilde{x}|^{\{\beta\}^+}}, \\ \text{if } \{\beta\}^+ &\in (0, 1), \\ |u|_\beta &= \sum_{|\gamma| \leq [\beta]^-} \sup_{(t,x) \in H} |\partial_x^\gamma u(t,x)| \\ &\quad + \sup_{|\gamma| = [\beta]^-, t, x, h \neq 0} \frac{|\partial_x^\gamma u(t, x+h) + \partial_x^\gamma u(t, x-h) - 2\partial_x^\gamma u(t, x)|}{|h|^{\{\beta\}^+}}, \\ \text{if } \{\beta\}^+ &= 1, \end{aligned}$$

is finite. Accordingly,  $C^\beta(\mathbf{R}^d)$  denotes the corresponding space of functions on  $\mathbf{R}^d$ . The classes  $C^\beta$  are Hölder-Zygmund spaces: they coincide with Hölder spaces if  $\beta \notin \mathbf{N}$  (see 1.2.2 of [24]).

Define for  $\beta = [\beta] + \{\beta\} > 0$  with  $[\beta] \in \mathbf{N}$ ,  $\{\beta\} \in (0, 1)$ ,

$$\begin{aligned} M_\beta^{(\alpha)} &= \mathbf{1}_{\{\alpha \in (0,2)\}} |c|_\beta + \mathbf{1}_{\{\alpha \in [1,2]\}} |a|_\beta + \mathbf{1}_{\{\alpha=2\}} |B|_\beta \\ &\quad + \mathbf{1}_{\{\alpha \in (0,2)\}} \sup_{|\gamma| \leq d_0, |y|=1} |\partial_y^\gamma m^{(\alpha)}(\cdot, y)|_\beta. \end{aligned} \quad (4)$$

We make the following assumptions.

**A1** (i) There is a constant  $\mu > 0$  such that for all  $x \in \mathbf{R}^d$  and  $|\xi| = 1$ ,

$$\begin{aligned} (B(x)\xi, \xi) &\geq \mu, \text{ if } \alpha = 2, \\ \int_{S^{d-1}} |(w, \xi)|^\alpha m_\alpha(x, w) d\xi &\geq \mu, \text{ if } \alpha \in (0, 2), \end{aligned} \quad (5)$$

where  $B(x) = b(x)^* b(x)$ ,  $x \in \mathbf{R}^d$ ;

(ii) It holds that

$$\lim_{n \rightarrow \infty} \sup_x \int_{U_n} |l_\alpha(x, v)|^\alpha \pi(dv) = 0, \text{ if } \alpha \in (0, 2].$$

**A2**( $\beta$ ) It satisfies that  $M_\beta^{(\alpha)} < \infty$ ,  $\inf_x |\det c(x)| > 0$ , and

$$\begin{aligned} &\int \left\{ \mathbf{1}_{U_1}(v) \left[ |l_\alpha(x, v)|^\alpha + \mathbf{1}_{\{\beta \geq 1\}} \sum_{j=1}^{[\beta]} (|\partial^j l_\alpha(x, v)|^\alpha + |\partial^j l_\alpha(x, v)|^{\frac{[\beta]}{j} \vee \alpha}) \right] \right. \\ &\quad \left. + \mathbf{1}_{U_1^c}(v) \left[ |l_\alpha(x, v)|^{\alpha \wedge 1} \wedge 1 + \mathbf{1}_{\{\beta \geq 1\}} \sum_{j=1}^{[\beta]} (|\partial^j l_\alpha(x, v)| + |\partial^j l^{(\alpha)}(x, v)|^{\frac{[\beta]}{j}}) \right] \right\} \pi(dv) \\ &\leq K, \end{aligned}$$

**A3**( $\beta$ ) For all  $x, x' \in \mathbf{R}^d$ ,

$$\int_{U_1} [|l_\alpha(x, v) - l_\alpha(x', v)|^\alpha + |\partial^{[\beta]} l_\alpha(x, v) - \partial^{[\beta]} l_\alpha(x', v)|^\alpha] \pi(dv) \leq C|x - x'|^{\alpha\beta}, \alpha \in [1, 2],$$

There exists  $\beta'$  such that  $\beta \leq \alpha + \beta' < \alpha + \beta$  and for all  $x, x' \in \mathbf{R}^d$ ,

$$\begin{aligned} & \mathbf{1}_{\{\beta \geq 1\}} \int_{U_1} (|l_\alpha(x, v) - l_\alpha(x', v)|^{(\alpha + \beta' - [\beta]) \wedge 1} \wedge 1) \\ & \times \sum_{j=1}^{[\beta]} (|\partial^j l_\alpha(x, v)|^{\alpha \vee 1} + |\partial^j l_\alpha(x, v)|^{\frac{[\beta]}{j} \vee \alpha}) \pi(dv) \\ & \leq C|x - x'|^{\beta - [\beta]}, \\ & \int_{U_1^c} [|l_\alpha(x, v) - l_\alpha(x', v)|^{(\alpha + \beta' - [\beta]) \wedge 1} \wedge 1] \\ & \times [1 + \mathbf{1}_{\{\beta \geq 1\}} \sum_{j=1}^{[\beta]} (|\partial^j l_\alpha(x, v)| + |\partial^j l_\alpha(x, v)|^{\frac{[\beta]}{j}})] \pi(dv) \\ & \leq C|x - x'|^{\beta - [\beta]}. \end{aligned}$$

**A4**( $\beta$ ) For  $\beta \geq 1, x, x' \in \mathbf{R}^d$ ,

$$\begin{aligned} & \mathbf{1}_{\{\beta \geq 1\}} \sum_{j=1}^{[\beta]} \int_{U_1^c} |\partial^j l_\alpha(x, v) - \partial^j l_\alpha(x', v)| \pi(dv) \\ & + \left( \int_{U_1^c} |\partial^j l_\alpha(x, v) - \partial^j l_\alpha(x', v)|^{\frac{[\beta]}{j}} \pi(dv) \right)^{\frac{j}{[\beta]}} \\ & \leq C|x - x'|^{\beta - [\beta]}, \end{aligned}$$

and

$$\begin{aligned} & \mathbf{1}_{\{\beta \geq 1\}} \sum_{j=1}^{[\beta]} \left( \int_{U_1} |\partial^j l_\alpha(x, v) - \partial^j l_\alpha(x', v)|^{\alpha \vee 1} \pi(dv) \right)^{\frac{1}{\alpha} \wedge 1} \\ & + \sum_{j=1}^{[\beta]} \left( \int_{U_1} |\partial^j l_\alpha(x, v) - \partial^j l_\alpha(x', v)|^{\frac{[\beta]}{j} \vee \alpha} \pi(dv) \right)^{\frac{j}{[\beta]} \wedge \frac{1}{\alpha}} \\ & \leq C|x - x'|^{\beta - [\beta]}. \end{aligned}$$

The main result of this paper is the following statement.

**Theorem 1.** *Let  $\alpha \in (0, 2]$ ,  $\beta > 0$ ,  $\beta \notin \mathbf{N}$ . Assume A1-A4( $\beta$ ) hold. Then there exists a constant  $C$  such that for all  $g \in C^{\alpha+\beta}(\mathbf{R}^d)$ ,  $f \in C^\beta(\mathbf{R}^d)$*

$$\begin{aligned} |\mathbf{E}g(Y_T) - \mathbf{E}g(X_T)| &\leq C|g|_{\alpha+\beta}\delta^{\kappa(\alpha,\beta)}, \\ |\mathbf{E}\int_0^T f(Y_{\tau_{i_s}})ds - \mathbf{E}\int_0^T f(X_s)ds| &\leq C|f|_\beta\delta^{\kappa(\alpha,\beta)}, \end{aligned} \quad (6)$$

where

$$\kappa(\alpha, \beta) = \begin{cases} \frac{\beta}{\alpha}, & \beta < \alpha, \\ 1, & \beta > \alpha. \end{cases}$$

**Remark 2.** 1. *The second condition of A1(i) holds with some constant  $\mu > 0$  if, for example, there is a Borel set  $\Gamma \subseteq S^{d-1}$  such that  $\mu_{d-1}(\Gamma) > 0$  and  $\inf_{x \in \mathbf{R}^d, w \in \Gamma} m_\alpha(x, w) > 0$ .*  
2. *The assumptions A1-A4( $\beta$ ) guarantee that the solution to the backward Kolmogorov equation associated with  $X_t$  is  $(\alpha + \beta)$ -Hölder. If  $\alpha = 2$  and the operator is differential, the assumptions imposed are standard classical. The regularity of the solution determines the rate of convergence of a weak Euler approximation.*

### 2.3. SDEs driven by Lévy processes

Let  $Z^0 = Z_t^0$ ,  $t \in [0, T]$ , be a standard  $d$ -dimensional spherically-symmetric  $\alpha$ -stable process (see (12) for the definition) with jump measure  $p_0$  and martingale measure  $q_0$ , and let  $Z_t = (Z_t^1, \dots, Z_t^m)$  be an independent  $m$ -dimensional Lévy process defined by

$$\begin{aligned} Z_t &= \int_0^t \int g_\alpha(v) p(ds, dv), \alpha \in (0, 1), \\ Z_t &= \int_0^t \int_{U_1} g_\alpha(v) q(ds, dv) + \int_0^t \int_{U_1^c} g_\alpha(v) p(ds, dv), \alpha = 1, \\ Z_t &= \int_0^t \int g_\alpha(v) q(ds, dv), \alpha \in (1, 2], \end{aligned} \quad (7)$$

where  $g_\alpha = (g_\alpha^i)_{1 \leq i \leq m}$  is a measurable function on  $U$  and

$$\int_{U_1} |g_\alpha(v)|^\alpha \pi(dv) + \mathbf{1}_{\{\alpha \in (1, 2)\}} \int_{U_1^c} |g_\alpha(v)| \pi(dv) < \infty.$$

Consider for  $t \in [0, T]$ ,

$$\begin{aligned} X_t &= X_0 + \int_0^t c(X_{s-}) dZ_s^0 + \int_0^t C(X_{s-}) dZ_s, t \in [0, T], \alpha \in (0, 2), \\ X_t &= X_0 + \int_0^t a_2(X_s) ds + \int_0^t b(X_s) dW_s + \int_0^t C(X_{s-}) dZ_s, \alpha = 2, \end{aligned} \quad (8)$$



where  $c(x) = (c^{ij}(x))_{1 \leq i, j \leq d}$ ,  $C(x) = (C^{ij}(x))_{1 \leq i \leq d, 1 \leq j \leq m}$ ,  $x \in \mathbf{R}^d$ , are measurable and bounded. Assume that  $c$  is non-degenerate with  $\inf_x \det |c(x)| > 0$ . Obviously, (8) can be rewritten as

$$\begin{aligned}
X_t &= X_0 + \int_0^t \int_{\mathbf{R}_0^d} c(X_{s-}) y p_0(ds, dy) \\
&\quad + \int_0^t \int C(X_{s-}) g_\alpha(v) p(ds, dv) \text{ if } \alpha \in (0, 1), \\
X_t &= X_0 + \int_0^t \int_{|y|>1} c(X_{s-}) y p_0(ds, dy) + \int_0^t \int_{|y|\leq 1} c(X_{s-}) y q_0(ds, dy) \\
&\quad + \int_0^t \int_{U_1} C(X_{s-}) g_\alpha(v) q(ds, dv) + \int_0^t \int_{U_1^c} C(X_{s-}) g_\alpha(v) p(ds, dv) \text{ if } \alpha = 1, \\
X_t &= X_0 + \int_0^t \int_{\mathbf{R}_0^d} c(X_{s-}) y q_0(ds, dy) \\
&\quad + \int_0^t \int_{U_1} C(X_{s-}) g_\alpha(v) q(ds, dv) + \int_0^t \int_{U_1^c} C(X_{s-}) g_\alpha(v) p(ds, dv) \text{ if } \alpha \in (1, 2), \\
X_t &= X_0 + \int_0^t a(X_s) ds + \int_0^t b(X_s) dW_s \\
&\quad + \int_0^t \int_{U_1} C(X_{s-}) g_\alpha(v) q(ds, dv) + \int_0^t \int_{U_1^c} C(X_{s-}) g_\alpha(v) p(ds, dv) \text{ if } \alpha = 2,
\end{aligned}$$

Applying Theorem 1 to (8) we obtain easily the following statements.

**Proposition 3.** *Let  $X_t, t \in [0, T]$  satisfy (8),  $\inf_{x \in \mathbf{R}^d} |\det c(x)| > 0$ ,  $\inf_{x \in \mathbf{R}^d} |\det b(x)| > 0$ .*

*For  $\alpha \in (0, 1)$ , we assume  $\beta \in (0, 1), \alpha + \beta > 1, c^{ij}, C^{ij} \in C^\beta(\mathbf{R}^d)$  and*

$$\int [|g_\alpha(v)| + |g_\alpha(v)|^\alpha] \pi(dv) < \infty.$$

*For  $\alpha \in [1, 2)$ , we assume  $\beta \neq 1, \beta < 2, a, c^{ij}, C^{ij} \in C^\beta(\mathbf{R}^d)$  and*

$$\int_{U_1} [|g_\alpha(v)|^\alpha + |g_\alpha(v)|^{\alpha+[\beta]}] \pi(dv) + \int_{U_1^c} [|g_\alpha(v)| + |g_\alpha(v)|^{1+[\beta]}] \pi(dv) < \infty.$$

*For  $\alpha = 2$ , we assume  $\beta < 3, \beta \notin \mathbf{N}, a, b, C^{ij} \in C^\beta(\mathbf{R}^d)$  and*

$$\int_{U_1} [|g_\alpha(v)|^2 + |g_\alpha(v)|^{2+[\beta]}] \pi(dv) + \int_{U_1^c} [|g_\alpha(v)| + |g_\alpha(v)|^{1+[\beta]}] \pi(dv) < \infty.$$

Then there exists a constant  $C$  such that for all  $g \in C^{\alpha+\beta}(\mathbf{R}^d)$ ,  $f \in C^\beta(\mathbf{R}^d)$

$$\begin{aligned} |\mathbf{E}g(Y_T) - \mathbf{E}g(X_T)| &\leq C|g|_{\alpha+\beta}\delta_\alpha^{\frac{\beta}{\alpha}\wedge 1}, \\ |\mathbf{E} \int_0^T f(Y_{\tau_{i_s}})ds - \mathbf{E} \int_0^T f(X_s)ds| &\leq C|f|_\beta\delta_\alpha^{\frac{\beta}{\alpha}\wedge 1}. \end{aligned}$$

**Remark 4.** Proposition 3 improves the rate of convergence for diffusion processes in [11] with  $\beta \in (1, 2)$ . Under the assumption of Proposition 3 (with  $\alpha = 2, C^{ij} = 0$ ), it was derived in [11] that the convergence rate is of the order  $\frac{1}{3-\beta} < \kappa(2, \beta) = \frac{\beta}{2}$  if  $\beta \in (1, 2)$ .

### 3. Backward Kolmogorov Equation

To determine the form of the backward Kolmogorov equation associated with  $X_t$  in (1), we find the compensator of the jump measure of  $X$  first.

**Lemma 5.** Let  $p^X$  be the jump measure of  $X_t$  in (1). Then

$$q^X(dt, dy) = p^X(dt, dy) = \tilde{m}_\alpha(X_{t-}, \frac{y}{|y|}) \frac{dy}{|y|^{d+\alpha}} dt + \int_U \mathbf{1}_{dy}(l_\alpha(X_{t-}, v)) \pi(dv) dt$$

is a martingale measure, where

$$\tilde{m}_\alpha(x, \frac{y}{|y|}) = \frac{1}{|\det c(x)|} \frac{1}{|c(x)^{-1} \frac{y}{|y|}|^{d+\alpha}} m_\alpha(x, \frac{c(x)^{-1} \frac{y}{|y|}}{|c(x)^{-1} \frac{y}{|y|}|}), x \in \mathbf{R}^d, y \in \mathbf{R}_0^d.$$

**Proof.** Since  $p_0$  and  $p$  have no common jumps, for any  $t$  and  $\Gamma \in \mathcal{B}(\mathbf{R}_0^d)$ ,

$$\begin{aligned} p^X((0, t] \times \Gamma) &= \sum_{s \leq t} \mathbf{1}_\Gamma(\Delta X_t) = \int_0^t \int_{\mathbf{R}_0^d} \mathbf{1}_\Gamma(c(X_{s-}) h_\alpha(X_{s-}, \frac{y}{|y|}) y) p_0(ds, dy) \\ &\quad + \int_0^t \int_U \mathbf{1}_\Gamma(l_\alpha(X_{s-}, v)) p(ds, dv), \end{aligned}$$

with  $\Delta X_t = X_t - X_{t-}$ ,  $0 < t$ ,  $\Gamma \in \mathcal{B}(\mathbf{R}_0^d)$ . Passing to polar coordinates and changing the variable of integration twice

$$\begin{aligned} &\int_{\mathbf{R}_0^d} \mathbf{1}_\Gamma(c(x) h_\alpha(x, \frac{y}{|y|}) y) \frac{dy}{|y|^{d+\alpha}} \\ &= \int_0^\infty \int_{S^{d-1}} \mathbf{1}_\Gamma(c(x) h_\alpha(x, w) \rho w) \mu_{d-1}(dw) \frac{d\rho}{\rho^{1+\alpha}} \\ &= \int_0^\infty \int_{S^{d-1}} \mathbf{1}_\Gamma(c(x) \rho w) \mu_{d-1}(dw) \frac{h_\alpha(x, w)^\alpha d\rho}{\rho^{1+\alpha}} \\ &= \int_{\mathbf{R}_0^d} \mathbf{1}_\Gamma(c(x) y) \frac{h_\alpha(x, \frac{y}{|y|})^\alpha dy}{|y|^{d+\alpha}} = \int_{\mathbf{R}_0^d} \mathbf{1}_\Gamma(y) \tilde{m}_\alpha(x, \frac{y}{|y|}) \frac{dy}{|y|^{d+\alpha}}. \end{aligned}$$

The statement follows. ■

For  $u \in C^{\alpha+\beta}(H)$ , denote

$$A_y^{(\alpha)}u(t, x) = u(t, x + y) - u(t, x) - \chi^{(\alpha)}(y)(\nabla_x u(t, x), y),$$

where  $\chi^{(\alpha)}(y) = \mathbf{1}_{\{|y| \leq 1\}} \mathbf{1}_{\{\alpha=1\}} + \mathbf{1}_{\{\alpha \in (1,2)\}}$ . Let

$$\begin{aligned} \mathcal{A}_z^{(\alpha)}u(t, x) &= \mathbf{1}_{\{\alpha=1\}}(a_1(z), \nabla_x u(t, x)) + \frac{1}{2} \mathbf{1}_{\{\alpha=2\}} \sum_{i,j=1}^d B^{ij}(z) \partial_{ij}^2 u(t, x) \\ &\quad + \int_{\mathbf{R}_0^d} A_y^{(\alpha)}u(t, x) \tilde{m}_\alpha(z, \frac{y}{|y|}) \frac{dy}{|y|^{d+\alpha}}, \quad x, z \in \mathbf{R}^d, \\ \mathcal{A}^{(\alpha)}u(t, x) &= \mathcal{A}_x^{(\alpha)}u(t, x) = \mathcal{A}_z^{(\alpha)}u(t, x)|_{z=x}, x \in \mathbf{R}^d. \end{aligned} \quad (9)$$

Let

$$\begin{aligned} \mathcal{B}_z^{(\alpha)}u(t, x) &= \mathbf{1}_{\{\alpha \in (1,2)\}}(a(z), \nabla_x u(t, x)) + \int_U [u(t, x + l_\alpha(z, v)) - u(t, x) \\ &\quad - \mathbf{1}_{\{\alpha \in (1,2)\}} \mathbf{1}_{U_1}(v)(\nabla_x u(t, x), l_\alpha(z, v))] \pi(dv), \\ \mathcal{B}^{(\alpha)}u(t, x) &= \mathcal{B}_x^{(\alpha)}u(t, x) = \mathcal{B}_z^{(\alpha)}u(t, x)|_{z=x}, x \in \mathbf{R}^d. \end{aligned} \quad (10)$$

**Remark 6.** Under assumptions A1-A4( $\beta$ ), for any  $\beta > 0$ , there exists a unique weak solution to equation (1) and for every  $u \in C^{\alpha+\beta}(\mathbf{R}^d)$ , the stochastic process

$$u(X_t) - \int_0^t (\mathcal{A}^{(\alpha)} + \mathcal{B}^{(\alpha)})u(X_s) ds \quad (11)$$

is a martingale (see [13]). The operator  $\mathcal{L}^{(\alpha)} = \mathcal{A}^{(\alpha)} + \mathcal{B}^{(\alpha)}$  is the generator of  $X_t$  defined in (1);  $\mathcal{A}^{(\alpha)}$  is the principal part of  $\mathcal{L}^{(\alpha)}$  and  $\mathcal{B}^{(\alpha)}$  is the lower order or subordinated part of  $\mathcal{L}^{(\alpha)}$ .

**Remark 7.** If  $m^{(\alpha)} = 1$ ,  $(B^{ij}) = I$  ( $d \times d$ -identity matrix),  $a_1(z) = 0$ , then  $\mathcal{A}^{(\alpha)}$  is the generator of a standard spherically-symmetric  $\alpha$ -stable process

$$\begin{aligned} Z_t &= \int_0^t \int y q^Z(ds, dy), \alpha \in (1, 2), \\ Z_t &= \int_0^t \int_{|y| \leq 1} y q^Z(ds, dy) + \int_0^t \int_{|y| > 1} y p^Z(ds, dy), \alpha = 1, \\ Z_t &= \int_0^t \int y p^Z(ds, dy), \alpha \in (0, 1), \end{aligned} \quad (12)$$

where  $p^Z(ds, dy)$  is the jump measure of  $Z$  and

$$q^Z(ds, dy) = p^Z(ds, dy) - \frac{dy ds}{|y|^{d+\alpha}}$$

is the martingale measure;  $Z_t$  is the standard Wiener process if  $\alpha = 2$ .

We consider in Hölder-Zygmund spaces the backward Kolmogorov equation associated with  $X_t$  (see [21], [15]):

$$\begin{aligned} (\partial_t + \mathcal{A}_x^{(\alpha)} + \mathcal{B}_x^{(\alpha)} - \lambda)v(t, x) &= f(t, x), \\ v(T, x) &= 0 \end{aligned} \quad (13)$$

with  $\lambda \geq 0$ . The regularity of its solution is essential for the one step estimate that determines the rate of convergence.

**Definition 8.** Let  $f$  be a bounded measurable function on  $\mathbf{R}^d$ . We say that  $u \in C^{\alpha+\beta}(H)$  is a solution to (13), if for each  $(t, x) \in H$ ,

$$u(t, x) = \int_0^t [(\partial_t + \mathcal{L}^{(\alpha)} - \lambda)u(s, x) - \lambda u(s, x) + f(s, x)] ds. \quad (14)$$

**Theorem 9.** Let  $\alpha \in (0, 2]$ ,  $\beta > 0, \beta \notin \mathbf{N}$ , and  $f \in C^\beta(H)$ . Assume A1-A4( $\beta$ ) hold. Then there exists a unique solution  $v \in C^{\alpha+\beta}(H)$  to (13). Moreover, there is a constant  $C$  independent of  $f$  such that

$$|u|_{\alpha+\beta} \leq C|f|_\beta.$$

An immediate consequence of this theorem is the following statement.

**Corollary 10.** Let  $\alpha \in (0, 2]$  and  $\beta > 0, \beta \notin \mathbf{N}$ . Assume A1-A4( $\beta$ ) hold,  $f \in C^\beta(H)$ , and  $g \in C^{\alpha+\beta}(\mathbf{R}^d)$ . Then there exists a unique solution  $v \in C^{\alpha+\beta}(H)$  to the Cauchy problem

$$\begin{aligned} (\partial_t + \mathcal{A}_x^{(\alpha)} + \mathcal{B}_x^{(\alpha)})v(t, x) &= f(x), \\ v(T, x) &= g(x), \end{aligned} \quad (15)$$

and  $|v|_{\alpha+\beta} \leq C(|f|_\beta + |g|_{\alpha+\beta})$  with a constant  $C$  independent of  $f$  and  $g$ .

To prove Theorem 9 and Corollary 10, we first derive Hölder norm estimates of  $\mathcal{A}^{(\alpha)}f$  and  $\mathcal{B}^{(\alpha)}f$ ,  $f \in C^{\alpha+\beta}(\mathbf{R}^d)$ ,  $\beta > 0$ , and an auxiliary lemma about uniform convergence of Hölder functions.

### 3.1. Kolmogorov Equation with Constant Coefficients

Let  $B = (B^{ij})_{1 \leq i, j \leq d}$  be a non-negative definite non-degenerate matrix. Let  $r_\alpha(y)$  be homogeneous with index zero and differentiable in  $y$  up to the order  $d_0 = [d/2] + 1$  and

$$\int_{S^{d-1}} wr_1(w) \mu_{d-1}(dw) = 0, r_2 = 0.$$

Let

$$\begin{aligned} A_\alpha^0 u(x) &= \mathbf{1}_{\{\alpha=2\}} B^{ij} \partial_{ij}^2 u(x) + \mathbf{1}_{\{\alpha=1\}} a_1^i \partial_i u(x) + \int_{\mathbf{R}^d} [u(x+y) - u(x) \\ &\quad - (\mathbf{1}_{\{|y| \leq 1\}} \mathbf{1}_{\{\alpha=1\}} + \mathbf{1}_{\{1 < \alpha < 2\}})(\nabla u(x), y)] r_\alpha(y) \frac{dy}{|y|^{d+\alpha}}. \end{aligned}$$

In terms of Fourier transform,

$$A_\alpha^0 u(x) = \mathcal{F}^{-1} [\psi_\alpha^0(\xi) \mathcal{F}u(\xi)](x),$$

where

$$\begin{aligned} \psi_\alpha^0(\xi) &= -N \int_{S^{d-1}} |(w, \xi)|^\alpha [1 - i(\mathbf{1}_{\{\alpha \neq 1\}} \tan \frac{\alpha\pi}{2} \text{sgn}(w, \xi) \\ &\quad - \frac{2}{\pi} \mathbf{1}_{\{\alpha=1\}} \text{sgn}(w, \xi) \ln |(w, \xi)|)] r_\alpha(w) \mu_{d-1}(dw) \\ &\quad - i \mathbf{1}_{\{\alpha=1\}}(a_1, \xi) - \mathbf{1}_{\{\alpha=2\}}(B\xi, \xi), \end{aligned}$$

where  $a_1 \in \mathbf{R}^d$ . We will need the following assumptions.

**B.** (i) There is a constant  $\mu > 0$  such that for all  $|\xi| = 1$ ,

$$\begin{aligned} (B\xi, \xi) &\geq \mu, \text{ if } \alpha = 2, \\ \int_{S^{d-1}} |(w, \xi)|^\alpha r_\alpha(w) \mu_{d-1}(dw) &\geq \mu, \text{ if } \alpha \in (0, 2); \end{aligned}$$

(ii) There is a constant  $K$  such that

$$|a_1| + |B| + \sup_{|\gamma| \leq d_0, y \in \mathbf{R}^d} |\partial^\gamma r_\alpha(y)| \leq K.$$

Consider for  $\lambda \geq 0$  the Cauchy problem

$$\begin{aligned} \partial_t u(t, x) &= A_\alpha^0 u(t, x) - \lambda u(t, x) + f(x), \quad (t, x) \in H \\ u(0, x) &= 0, \quad x \in \mathbf{R}^d. \end{aligned} \tag{16}$$

We will solve this equation for  $f \in C_b^\infty(\mathbf{R}^d)$  and pass to the limit. The following approximation statement is needed.

**Lemma 11.** *Let  $\beta > 0, f \in C^\beta(\mathbf{R}^d)$ . Then there is a sequence  $f_n \in C_b^\infty(\mathbf{R}^d)$  such that*

$$|f_n|_\beta \leq 2|f|_\beta, |f|_\beta \leq \liminf_n |f_n|_\beta,$$

*and for any  $0 < \beta' < \beta$ ,*

$$|f_n - f|_{\beta'} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

**Proof.** By Lemma 6.1.7 in [2], there exists a function  $\phi \in C_0^\infty(\mathbf{R}^d)$  such that  $\text{supp}\phi(\xi) = \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}, \phi(\xi) > 0$ , if  $2^{-1} < |\xi| < 2$ , and

$$\sum_{j=-\infty}^{\infty} \phi(2^{-j}\xi) = 1, \text{ if } \xi \neq 0.$$

Define functions  $\varphi_k \in \mathcal{S}(\mathbf{R}^d), k = 1, 2, \dots$  by

$$\mathcal{F}\varphi_k = \phi(2^{-k}\xi), \quad (17)$$

and  $\varphi_0 \in \mathcal{S}(\mathbf{R}^d)$  by

$$\mathcal{F}\varphi_0 = 1 - \sum_{k \geq 1} \varphi_k(\xi). \quad (18)$$

We will use on  $C^\beta(\mathbf{R}^d)$  an equivalent norm (see 2.3.8 and 2.3.1 in [23])

$$|f|_{\beta; \infty\infty} = \sup_{k \geq 0, x \in \mathbf{R}^d} 2^{\beta k} |\varphi_k * f(x)|.$$

Obviously,  $f_n \in C_b^\infty(\mathbf{R}^d)$ . Let

$$f_n = \sum_{k=0}^n \varphi_k * f, n \geq 1.$$

Since

$$\varphi_k = \sum_{l=-1}^1 \varphi_{k+l} * \varphi_k, \varphi_0 = (\varphi_0 + \varphi_1) * \varphi_0,$$

we have for large  $n$ ,

$$\begin{aligned} f_n * \varphi_k &= f * \varphi_k, k < n, \\ f_n * \varphi_n &= f * \varphi_n - f * \varphi_{n+1} * \varphi_n, \\ f_n * \varphi_{n+1} &= f * \varphi_n * \varphi_{n+1}, f_n * \varphi_k = 0, k > n+1, \end{aligned}$$

and the statement follows.

$$|f_n|_\beta \leq 2|f|_{\alpha,\beta}, |f|_\beta \leq \liminf_n |f_n|_\beta,$$

and for any  $0 < \beta' < \beta$

$$\begin{aligned} |f_n - f|_{\beta'; \infty\infty} &\leq 2 \sup_{k \geq n} \sup_x 2^{\beta' k} |\varphi_k * f(x)| \\ &\leq 2 \cdot 2^{-(\beta-\beta')n} \sup_{k \geq n} \sup_x 2^{\beta k} |\varphi_k * f(x)| \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . ■

**Proposition 12.** *Let  $\alpha \in (0, 2]$ ,  $\beta > 0, \beta \notin \mathbf{N}, f \in C^\beta(\mathbf{R}^d)$ . Assume B holds. Then there is a unique solution  $u \in C^{\alpha+\beta}(H)$  to (16) and*

$$|u|_{\alpha+\beta} \leq C|f|_\beta,$$

where the constant  $C$  depends only on  $\alpha, \beta, T, d, \mu, K$ . Moreover,

$$|u|_\beta \leq C(\alpha, d)(\lambda^{-1} \wedge T)|f|_\beta,$$

and there is a constant  $C$  such that for all  $s \leq t \leq T$ ,

$$|u(t, \cdot) - u(s, \cdot)|_{\alpha/2+\beta} \leq C(t-s)^{1/2}|f|_\beta.$$

**Proof.** By Lemma 11 there is a sequence  $f_n \in C_b^\infty(\mathbf{R}^d)$  such that

$$|f_n|_\beta \leq 2|f|_\beta, |f|_\beta \leq \liminf_n |f_n|_\beta,$$

and for any  $\beta' < \beta$

$$|f_n - f|_{\beta'} \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (19)$$

Then, by Lemma 7 in [14] for each  $n$  there is a unique  $u_n \in C_b^\infty(H)$  solving (16). Moreover,

$$u_n(t, x) = R_\lambda f_n(t, x) = \int_0^t G_{s,t}^\lambda * f_n(x) ds,$$

where

$$G_{s,t}^\lambda(x) = \mathcal{F}^{-1} \exp \left\{ \int_s^t (\psi_\alpha^0(r, \xi) - \lambda) dr \right\}, 0 \leq s \leq t \leq T.$$

Since for any  $k \leq [\beta]$ ,

$$\partial^k u_n(t, x) = \int_0^t G_{s,t}^\lambda * \partial^k f_n(x) ds = R_\lambda(\partial^k f_n),$$

it follows by Lemma 17 in [14] that for every  $\beta' \in ([\beta], \beta]$  there is a constant  $C$  depending only on  $\alpha, \beta', p, T, d, \mu, K$  such that

$$|\partial^k u_n|_{\alpha+\beta'-[\beta]} \leq C |\partial^k f_n|_{\beta'-[\beta]} \quad (20)$$

for all  $k \leq [\beta]$ . Moreover,

$$|\partial^k u_n|_{\beta'-[\beta]} \leq c(\alpha, d)(\lambda^{-1} \wedge T) |\partial^k f_n|_{\beta-[\beta]}$$

and there is a constant  $C$  such that for all  $s \leq t \leq T$ ,

$$|\partial^k u_n(t, \cdot) - u_n(s, \cdot)|_{\alpha/2+\beta} \leq C(t-s)^{1/2} |\partial^k f_n|_\beta$$

for all  $k \leq [\beta]$ . Let  $[\beta] < \beta' < \beta$ . Then, there is a constant  $C$  depending only on  $\alpha, \beta', T, d, \mu, K$  such that

$$|u_n|_{\alpha+\beta'-[\beta]} \leq C |f_n|_{\beta'-[\beta]}. \quad (21)$$

Moreover,

$$|u_n|_{\beta-[\beta]} \leq c(\alpha, d)(\lambda^{-1} \wedge T) |f_n|_{\beta-[\beta]} \quad (22)$$

and there is a constant  $C$  such that for all  $s \leq t \leq T$ ,

$$|u_n(t, \cdot) - u_n(s, \cdot)|_{\alpha/2+\beta} \leq C(t-s)^{1/2} |f_n|_\beta. \quad (23)$$

By Lemma 11 and (19), there exists  $u \in C^{\alpha+\beta'}(H)$  such that  $u_n \rightarrow u$  in  $C^{\alpha+\beta'}(H)$ . Therefore  $u$  satisfies (14) with  $A_\alpha^0$  instead of  $\mathcal{L}^{(\alpha)}$ . Since (20) holds with  $\beta' = \beta$ , the solution  $u \in C^{\alpha+\beta}(H)$  and the statement is proved.  $\blacksquare$

### 3.1.1. Estimates of $\mathcal{B}^{(\alpha)} f, f \in C^{\alpha+\beta}$

We will use the following equality for the estimates of  $\mathcal{B}^{(\alpha)}$ .

**Lemma 13.** (Lemma 2.1 in [7]) For  $\delta \in (0, 1)$  and  $u \in C_0^\infty(\mathbf{R}^d)$ ,

$$u(x+y) - u(x) = K \int k^{(\delta)}(y, z) \partial^\delta u(x-z) dz, \quad (24)$$

where  $K = K(\delta, d)$  is a constant,

$$k^{(\delta)}(y, z) = |z+y|^{-d+\delta} - |z|^{-d+\delta},$$

and there exists a constant  $C$  such that

$$\int |k^{(\delta)}(y, z)| dz \leq C |y|^\delta, \forall y \in \mathbf{R}^d.$$



First we prove the following auxiliary estimate.

**Lemma 14.** *Let  $\lambda \geq 1, \eta(dv)$  be a nonnegative measure on  $U$  and let  $\mu \in \mathbf{N}_0^d$  be a multiindex such that  $n \geq |\mu|$ , and  $\sum_{j=1}^k \gamma_j = \mu$  with  $\gamma_j \in \mathbf{N}_0^d, \gamma_j \neq 0, k \geq \lambda$ . Then there exist numbers  $\theta(\lambda, j) \in [0, 1]$  and a constant  $C$  such that for any nonnegative measurable functions  $l_{\lambda, \gamma_j}$  on  $U$ ,*

$$\begin{aligned} \int_U \prod_j |l_{\lambda, \gamma_j}| d\eta &\leq C \prod_j \left( \int_{U_1} |l_{\lambda, \gamma_j}|^{\frac{n}{|\gamma_j|} \vee \lambda} d\eta \right)^{\left(\frac{|\gamma_j|}{n} \wedge \lambda\right) \theta(\lambda, j)} \left( \int_U |l_{\lambda, \gamma_j}|^\lambda d\eta \right)^{\frac{1}{\lambda} (1 - \theta(\lambda, j))} \\ &\leq C \prod_j \left\{ \left( \int_{U_1} |l_{\lambda, \gamma_j}|^{\frac{n}{|\gamma_j|} \vee \lambda} d\eta \right)^{\left(\frac{|\gamma_j|}{n} \wedge \lambda\right)} + \left( \int_U |l_{\lambda, \gamma_j}|^\lambda d\eta \right)^{\frac{1}{\lambda}} \right\}. \end{aligned}$$

In addition, there is a constant  $C$  such that

$$\int_U \prod_j |l_{\lambda, \gamma_j}| d\eta \leq C \sum_j \int_U [|l_{\lambda, \gamma_j}|^{\frac{n}{|\gamma_j|} \vee \lambda} + |l_{\lambda, \gamma_j}|^\lambda] d\eta.$$

**Proof.** If there is  $\gamma_{j_0}$  for which  $\frac{|\mu|}{|\gamma_{j_0}|} < \lambda$  or  $|\gamma_{j_0}| > \frac{|\mu|}{\lambda}$  ( $\lambda > 1$  in this case and there could be only one  $\gamma_{j_0}$  like this), then  $|\mu| - |\gamma_{j_0}| < |\mu|(1 - \frac{1}{\lambda})$  and for  $\gamma_j \neq \gamma_{j_0}$

$$\lambda \leq \frac{\lambda}{\lambda - 1} \frac{|\mu| - |\gamma_{j_0}|}{|\gamma_j|} \leq \frac{|\mu|}{|\gamma_j|} \leq \frac{n}{|\gamma_j|}.$$

By Hölder's inequality,

$$\begin{aligned} \int_U \prod_j |l_{\lambda, \gamma_j}| d\eta &\leq \left( \int_U |l_{\lambda, \gamma_{j_0}}|^\lambda d\eta \right)^{\frac{1}{\lambda}} \left( \int_U \prod_{j \neq j_0} |l_{\lambda, \gamma_j}|^{\frac{\lambda}{\lambda-1}} d\eta \right)^{1-\frac{1}{\lambda}} \\ &\leq C \left[ \int_U |l_{\lambda, \gamma_{j_0}}|^\lambda d\eta + \int_U \prod_{j \neq j_0} |l_{\lambda, \gamma_j}|^{\frac{\lambda}{\lambda-1}} d\eta \right], \end{aligned}$$

where  $\sum_{j \neq j_0} \gamma_j = \mu - \gamma_{j_0}$  and  $\sum_{j \neq j_0} \frac{|\gamma_j|}{|\mu| - |\gamma_{j_0}|} = 1$ . Hence, by Hölder's inequality,

$$\begin{aligned} \int_U \prod_{j \neq j_0} |l_{\lambda, \gamma_j}|^{\frac{\lambda}{\lambda-1}} d\eta &\leq \prod_{j \neq j_0} \left( \int_U |l_{\lambda, \gamma_j}|^{\frac{\lambda}{\lambda-1} \frac{|\mu| - |\gamma_{j_0}|}{|\gamma_j|}} d\eta \right)^{\frac{|\gamma_j|}{|\mu| - |\gamma_{j_0}|}} \\ &\leq C \sum_{j \neq j_0} \int_U |l_{\lambda, \gamma_j}|^{\frac{\lambda}{\lambda-1} \frac{|\mu| - |\gamma_{j_0}|}{|\gamma_j|}} d\eta \\ &\leq C \sum_{j \neq j_0} \int_U (|l_{\lambda, \gamma_j}|^\lambda + |l_{\lambda, \gamma_j}|^{\frac{n}{|\gamma_j|}}) d\eta, \end{aligned}$$

and by the interpolation inequality there are  $\theta(\lambda, j) \in [0, 1]$  such that

$$\begin{aligned} \int_U \prod_j |l_{\lambda, \gamma_j}| d\eta &\leq \left( \int_U |l_{\lambda, \gamma_{j_0}}|^\lambda d\pi \right)^{\frac{1}{\lambda}} \prod_{j \neq j_0} \left( \int_U |l_{\lambda, \gamma_j}|^{\frac{\lambda}{\lambda-1} \frac{|\mu| - |\gamma_{j_0}|}{|\gamma_j|}} d\pi \right)^{\frac{\lambda-1}{\lambda} \frac{|\gamma_j|}{|\mu| - |\gamma_{j_0}|}} \\ &\leq \prod_j \left( \int_U |l_{\lambda, \gamma_j}|^\lambda d\pi \right)^{\frac{1}{\lambda} (1 - \theta(\lambda, j))} \left( \int_U |l_{\lambda, \gamma_j}|^{\frac{n}{|\gamma_j|} \vee \lambda} d\pi \right)^{(\frac{|\gamma_j|}{n} \wedge \frac{1}{\lambda}) \theta(\lambda, j)}. \end{aligned}$$

If for all  $j$ ,  $\frac{|\mu|}{|\gamma_j|} \geq \lambda$ , then  $\sum_j \frac{|\gamma_j|}{|\mu|} = 1$  and by Hölder's inequality,

$$\begin{aligned} \int_U \prod_j |l_{\lambda, \gamma_j}| d\eta &\leq \prod_j \left( \int_U |l_{\lambda, \gamma_j}|^{\frac{|\mu|}{|\gamma_j|}} d\eta \right)^{\frac{|\gamma_j|}{|\mu|}} \\ &\leq C \sum_j \int_U |l_{\lambda, \gamma_j}|^{\frac{|\mu|}{|\gamma_j|}} d\eta \\ &\leq C \sum_j \int_{U_1} [|l_{\lambda, \gamma_j}|^{\frac{n}{|\gamma_j|}} + |l_{\lambda, \gamma_j}|^\lambda] d\eta. \end{aligned}$$

Also, by interpolation inequalities,

$$\prod_j \left( \int_U |l_{\lambda, \gamma_j}|^{\frac{|\mu|}{|\gamma_j|}} d\eta \right)^{\frac{|\gamma_j|}{|\mu|}} \leq \prod_j \left( \int_U |l_{\lambda, \gamma_j}|^\lambda d\eta \right)^{\frac{1}{\lambda} (1 - \theta(\lambda, j))} \left( \int_{U_1} |l_{\lambda, \gamma_j}|^{\frac{n}{|\gamma_j|}} d\eta \right)^{\frac{|\gamma_j|}{n} \theta(\lambda, j)}.$$

The statement follows. ■

**Proposition 15.** *Let  $\beta > 0, \beta \notin \mathbf{N}$ . Assume A1-A4( $\beta$ ) hold. Then for each  $\varepsilon > 0$  there exists a constant  $C_\varepsilon$  such that*

$$|\mathcal{B}^{(\alpha)} f|_\beta \leq \varepsilon |f|_{\alpha+\beta} + C_\varepsilon |f|_{\beta - [\beta]}, f \in C^{\alpha+\beta}(\mathbf{R}^d).$$

**Proof.** For  $\gamma \in \mathbf{N}_0^d, |\gamma| \leq [\beta], x \in \mathbf{R}^d$ ,

$$\begin{aligned} \partial^\gamma [\mathcal{B}_x^{(\alpha)} f(x)] &= \sum_{\nu + \mu = \gamma} \partial_z^\mu \mathcal{B}_z^{(\alpha)} \partial^\nu f(x)|_{z=x} \\ &= \mathcal{B}_x^{(\alpha)} \partial^\gamma f(x) + \sum_{\nu + \mu = \gamma, \mu \neq 0} \partial_z^\mu \mathcal{B}_z^{(\alpha)} \partial^\nu f(x)|_{z=x}. \end{aligned}$$

For  $\mu \neq 0$ ,

$$\begin{aligned} &\partial_z^\mu \mathcal{B}_z^{(\alpha)} \partial^\nu f(x)|_{z=x} \\ &= \mathbf{1}_{\{\alpha > 1\}} \int_{U_1} \partial_z^\mu [\partial^\nu f(x + l_\alpha(z, v)) - \partial^\nu f(x) - (\nabla f(x), l_\alpha(z, v))] |_{z=x} \pi(dv) \\ &\quad + \int \theta_\alpha(v) \partial_z^\mu \partial^\nu f(x + l_\alpha(z, v)) |_{z=x} \pi(dv) = T_1(x) + T_2(x), \end{aligned} \tag{25}$$

with  $\theta_\alpha(v) = \mathbf{1}_{\{\alpha \leq 1\}} + \mathbf{1}_{\{\alpha > 1\}} \mathbf{1}_{U_1^c}(v)$ , and

$$\begin{aligned}
& \mathcal{B}_x^{(\alpha)} \partial^\gamma f(x) \\
&= \int \theta_\alpha(v) [\partial^\gamma f(x + l_\alpha(x, v)) - \partial^\gamma f(x)] d\pi \\
&\quad + \mathbf{1}_{\{\alpha > 1\}} \int_{U_1} [\partial^\nu f(x + l_\alpha(x, v)) - \partial^\nu f(x) - (\nabla f(x), l_\alpha(x, v))] \pi(dv) \\
&= S_1(x) + S_2(x).
\end{aligned}$$

*Estimates of  $S_1$ .* For any  $\beta' \in ([\beta], \beta)$  there is a constant  $C$  such that

$$\left| \int \theta_\alpha(v) [\partial^\gamma f(x + l_\alpha(x, v)) - \partial^\gamma f(x)] d\pi \right| \leq C |f|_{\beta'} \int \theta_\alpha(v) |l_\alpha(x, v)|^{\alpha \wedge 1} \wedge 1 d\pi.$$

For  $x, x' \in \mathbf{R}^d$ ,

$$|S_1(x) - S_1(x')| \leq S_{11} + S_{12},$$

where

$$\begin{aligned}
S_{11} &= \int \theta_\alpha(v) |[\partial^\gamma f(x + l_\alpha(x, v)) - \partial^\gamma f(x)] \\
&\quad - [\partial^\gamma f(x' + l_\alpha(x, v)) - \partial^\gamma f(x')]| d\pi, \\
S_{12} &= \int \theta_\alpha(v) |\partial^\gamma f(x' + l_\alpha(x, v)) - \partial^\gamma f(x' + l_\alpha(x', v))| d\pi.
\end{aligned}$$

For  $\beta' < \beta$ , by assumption A3( $\beta$ ),

$$\begin{aligned}
S_{12} &\leq C |f|_{\beta'} \int \theta_\alpha(v) |\Delta l_\alpha(x, x', v)|^{(\alpha + \beta' - [\beta]) \wedge 1} \wedge 1 d\pi \\
&\leq C |x - x'|^{\beta - [\beta]},
\end{aligned}$$

with  $\Delta l_\alpha(x, x', v) = l_\alpha(x, v) - l_\alpha(x', v)$ . For each  $n$ , by Lemma 13,

$$\begin{aligned}
S_{11} &= \int_{U_n} \dots + \int_{U_n^c} \dots \\
&\leq \mathbf{1}_{\{\alpha < 1\}} \int_{U_n} ||\partial|^\alpha \partial^\gamma f(x - z) - \partial|^\alpha \partial^\gamma f(x - z)|| k^{(\alpha)}(l_\alpha(x, v), z) dz d\pi \\
&\quad + |f|^\beta |x - x'|^{\beta - [\beta]} \\
&\leq C [|f|_{\alpha + \beta} \mathbf{1}_{\{\alpha < 1\}} \sup_x \int_{U_n} l_\alpha(x, v)^\alpha d\pi + |f|^\beta] |x - x'|^{\beta - [\beta]}
\end{aligned}$$

■

**Proof.** *Estimates of  $S_2$ .* Let  $\alpha > 1$ . For  $g \in C^{\alpha+\beta-[\beta]}$ , denote

$$\begin{aligned}\mathcal{T}_h g(x) &= g(x+h) - g(x) - (\nabla g(x), h), \\ D_h \nabla g(x) &= \nabla g(x+h) - \nabla g(x), x, h \in \mathbf{R}^d.\end{aligned}$$

By lemma 13,

$$\begin{aligned}\mathcal{T}_h g(x) &= \int_0^1 (\nabla g(x+sh) - \nabla g(x), h) ds \\ &= \int_0^1 \int (|\partial|^{\alpha-1} \nabla g(x-z) k^{(\alpha-1)}(sh, z), h) dz ds, x, h \in \mathbf{R}^d.\end{aligned}\tag{26}$$

For  $\alpha > 1$ ,

$$\begin{aligned}S_2 &= \int_{U_1} \mathcal{T}_{l_\alpha(x,v)} \partial^\nu f(x) \pi(dv) \\ &= \int_{U_1} \int_0^1 (D_{sl_\alpha(x,v)} \nabla \partial^\nu f(x), l_\alpha(x, v)) ds \pi(dv)\end{aligned}$$

and for any  $\beta' \in ([\beta], \beta)$ ,

$$|S_2(x)| \leq C|f|_{\alpha+\beta'} \int |l_\alpha(x, v)|^\alpha d\pi \leq C|f|_{\alpha+\beta'}.$$

For  $x, x' \in \mathbf{R}^d, \alpha > 1$ ,

$$\begin{aligned}S_2(x) - S_2(x') &= \int_{U_1} [\mathcal{T}_{l_\alpha(x,v)} \partial^\nu f(x) - \mathcal{T}_{l_\alpha(x,v)} \partial^\nu f(x')] d\pi \\ &\quad + \int_{U_1} [\mathcal{T}_{l_\alpha(x,v)} \partial^\nu f(x') - \mathcal{T}_{l_\alpha(x',v)} \partial^\nu f(x')] d\pi \\ &= S_{21} + S_{22}.\end{aligned}$$

Since for any  $\beta' \in ([\beta], \beta)$ ,

$$\begin{aligned}&|T_{l_\alpha(x,v)} \partial^\nu f(x') - T_{l_\alpha(x',v)} \partial^\nu f(x')| \\ &\leq C|f|_{\alpha+\beta'} (|l_\alpha(x', v)|^{\alpha-1} + |l_\alpha(x, v)|^{\alpha-1}) |\Delta l_\alpha(x, x', v)|,\end{aligned}$$

then by Hölder's inequality,

$$\begin{aligned}|S_{22}| &\leq C|f|_{\alpha+\beta'} \left( \int_{U_1} |\Delta l_\alpha(x, x', v)|^\alpha d\pi \right)^{1/\alpha} \\ &\leq C|f|_{\alpha+\beta'} |\beta - \beta'|.\end{aligned}$$

By Lemma 13 and (26), for each  $n$  and  $\beta' \in ([\beta], \beta)$ , there is a constant  $C'$  such that

$$\begin{aligned}
|S_{21}| &\leq \int_{U_n} \int_0^1 \int |\partial^{\alpha-1} \nabla \partial^\nu f(x-z) - \partial^{\alpha-1} \nabla \partial^\nu f(x'-z)| \\
&\quad \times |k^{(\alpha-1)}(sl_\alpha(x, v), z)| |l_\alpha(x, v)| dz ds \pi(dv) \\
&\quad + \int_{U_1 \setminus U_n} |\mathcal{T}_{l_\alpha(x, v)} \partial^\nu f(x) - \mathcal{T}_{l_\alpha(x, v)} \partial^\nu f(x')| d\pi \\
&\leq C |f|_{\alpha+\beta} |x-x'|^{\beta-[\beta]} \int_{U_n} |l_\alpha(x, v)|^\alpha d\pi \\
&\quad + C' |f|_{\alpha+\beta'} |x-x'|^{\beta-[\beta]} \int_{U_1 \setminus U_n} (1 + |l_\alpha(x, v)|) d\pi.
\end{aligned}$$

*Estimates of  $T_1$ .* If  $\alpha > 1$ , then  $T_1(x) = A_1(x) + A_2(x)$ , where

$$A_1(x) = \int_{U_1} (\nabla \partial^\nu f(x + l_\alpha(z, v)) - \nabla \partial^\nu f(x), \partial_z^\mu l_\alpha(z, v))|_{z=x} d\pi, x \in \mathbf{R}^d,$$

and  $A_2(x)$  consists of the sum whose terms are of the form

$$\int_{U_1} \partial^{\nu+\kappa} f(x + l_\alpha) \prod_{\kappa_i \neq 0, j} \partial^{\gamma_j^i} l_\alpha^i d\pi$$

with the non-zero multiindices  $\gamma_j^i \in \mathbf{N}_0^d$  such that  $\sum_{\kappa_i \neq 0, j} \gamma_j^i = \mu$  and  $|\mu| \geq |\kappa| \geq 2$ .

Applying Hölder's inequality, we have

$$\begin{aligned}
|A_1(x)| &\leq C |f|_{\alpha+[\beta]} \int_{U_1} (|l_\alpha(x, v)| \wedge 1)^{\alpha-1} |\partial_z^\mu l_\alpha(x, v)| d\pi \\
&\leq C |f|_{\alpha+[\beta]} \left( \int_{U_1} (|l_\alpha(x, v)| \wedge 1)^\alpha d\pi \right)^{1-\frac{1}{\alpha}} \left( \int_{U_1} |\partial_z^\mu l_\alpha(x, v)|^\alpha d\pi \right)^{1/\alpha} \\
&\leq C |f|_{\alpha+[\beta]}. \tag{27}
\end{aligned}$$

Obviously,

$$\left| \int_{U_1} \partial^{\nu+\kappa} f(x + l_\alpha(x, v)) \prod_{\kappa_i \neq 0, j} \partial^{\gamma_j^i} l_\alpha^i(x, v) d\pi \right| \leq |f|_\beta \int_{U_1} \prod_{\kappa_i \neq 0, j} |\partial^{\gamma_j^i} l_\alpha^i(x, v)| d\pi.$$

By Lemma 14,

$$\int_{U_1} \prod_{\substack{\kappa_i \neq 0, \\ (i, j) \neq (i_0, j_0)}} |\partial^{\gamma_j^i} l_\alpha^i|^{\frac{\alpha}{\alpha-1}} d\pi \leq C \prod_{\kappa_i \neq 0, j} \left[ \left( \int_{U_1} |\partial^{\gamma_j^i} l_\alpha^i|^\alpha d\pi \right)^{\frac{1}{\alpha}} + \left( \int_{U_1} |\partial^{\gamma_j^i} l_\alpha^i|^{\frac{[\beta]}{|\gamma_j^i|}} d\pi \right)^{\frac{|\gamma_j^i|}{[\beta]}} \right].$$

Hence,  $|A_2(x)| \leq C|f|_\beta, x \in \mathbf{R}^d$ . Thus there exists  $\beta' < \beta$  such that  $|T_1(x)| \leq C|f|_{\beta'}, x \in \mathbf{R}^d$ .

Now we estimate the differences. For  $x, x' \in \mathbf{R}^d$  and a multiindex  $\sigma$ , denote  $\Delta \partial^\sigma l_\alpha(x, x'; v) = \partial^\sigma l_\alpha(x, v) - \partial^\sigma l_\alpha(x', v), v \in U$ . For any  $\beta' > [\beta] + 1$ ,

$$\begin{aligned} |A_1(x) - A_1(x')| &\leq |f|_{\beta'} \left[ \int_{U_1} (|\Delta l_\alpha(x, x'; v)| \wedge 1) |\partial_z^\mu l_\alpha(x, v)| d\pi \right. \\ &\quad \left. + \int_{U_1} (|l_\alpha(x', v)| \wedge 1) |\Delta \partial_z^\mu l_\alpha(x, x'; v)| d\pi \right] \\ &= |f|_{\beta'} [A_{11} + A_{12}]. \end{aligned}$$

Now

$$\begin{aligned} A_{12} &\leq \left( \int_{U_1} (|l_\alpha(x', v)| \wedge 1)^{\frac{\alpha}{\alpha-1}} d\pi \right)^{1-\frac{1}{\alpha}} \left( \int_{U_1} |\Delta \partial_z^\mu l_\alpha(x, x'; v)|^\alpha d\pi \right)^{\frac{1}{\alpha}} \\ &\leq \left( \int_{U_1} (|l_\alpha(x', v)| \wedge 1)^\alpha d\pi \right)^{1-\frac{1}{\alpha}} \left( \int_{U_1} |\Delta \partial_z^\mu l_\alpha(x, x'; v)|^\alpha d\pi \right)^{\frac{1}{\alpha}} \\ &\leq C|x - x'|^{\beta - [\beta]}. \end{aligned}$$

Obviously,

$$\begin{aligned} A_{11} &= \int_{U_1} (|\Delta l_\alpha(x, x'; v)| \wedge 1) |\partial^\mu l_\alpha(x, v)| d\pi \\ &\leq \int_{U_1} \mathbf{1}_{\{|\partial_z^\mu l_\alpha(x, v)| \leq 1\}} (|\Delta l_\alpha(x, x'; v)| \wedge 1) |\partial^\mu l_\alpha(x, v)| d\pi \\ &\quad + \int_{U_1} \mathbf{1}_{\{|\partial_z^\mu l_\alpha(x, v)| > 1\}} (|\Delta l_\alpha(x, x'; v)| \wedge 1) |\partial_z^\mu l_\alpha(x, v)| d\pi. \end{aligned}$$

By Hölder's inequality,

$$\begin{aligned} &\int_{|\partial_z^\mu l_\alpha(x, v)| \leq 1} \mathbf{1}_{U_1}(v) (|\Delta l_\alpha(x, x'; v)| \wedge 1) |\partial^\mu l_\alpha(x, v)| d\pi \\ &\leq \left( \int_{|\partial^\mu l_\alpha(x, v)| \leq 1} \mathbf{1}_{U_1}(v) (|\Delta l_\alpha(x, x'; v)| \wedge 1)^\alpha d\pi \right)^{\frac{1}{\alpha}} \\ &\quad \times \left( \int_{|\partial^\mu l_\alpha(x, v)| \leq 1} \mathbf{1}_{U_1}(v) |\partial_z^\mu l_\alpha(x, v)|^{\frac{\alpha}{\alpha-1}} d\pi \right)^{1-\frac{1}{\alpha}} \\ &\leq C|x - x'|^{\beta - [\beta]} \left( \int_{|\partial_z^\mu l_\alpha(x, v)| \leq 1} \mathbf{1}_{U_1}(v) |\partial^\mu l_\alpha(x, v)|^\alpha d\pi \right)^{1-\frac{1}{\alpha}} \\ &\leq C|x - x'|^{\beta - [\beta]}, \end{aligned}$$

and

$$\begin{aligned}
& \int_{|\partial^\mu l_\alpha(x,v)| > 1} \mathbf{1}_{U_1}(v) (|\Delta l_\alpha(x, x'; v)| \wedge 1) |\partial^\mu l_\alpha(x, v)| d\pi \\
& \leq \int_{|\partial^\mu l_\alpha(x,v)| > 1} \mathbf{1}_{U_1}(v) (|\Delta l_\alpha(x, x'; v)| \wedge 1) |\partial^\mu l_\alpha(x, v)|^\alpha d\pi \\
& \leq C |x - x'|^{\beta - [\beta]}.
\end{aligned}$$

Hence,

$$A_{11} \leq C |x - x'|^{\beta - [\beta]}, A_{12} \leq C |x - x'|^{\beta - [\beta]}.$$

Since  $A_2(x)$  consists of the sum whose terms are of the form

$$\int \partial^{\nu+\kappa} f(x + l_\alpha) \prod_{\kappa_i \neq 0, j} \partial^{\gamma_j^i} l_\alpha^i d\pi$$

with the non-zero multiindices  $\gamma_j^i \in \mathbf{N}_0^d$  such that  $\sum_{\kappa_i \neq 0, j} \gamma_j^i = \mu$  and  $|\mu| \geq |\kappa| \geq 2$ , we estimate the differences of a generic term

$$\tilde{A}_2(x) = \int \partial^{\nu+\kappa} f(x + l_\alpha(x, v)) \prod_{\kappa_i \neq 0, j} \partial^{\gamma_j^i} l_\alpha^i(x, v) d\pi.$$

We have

$$\begin{aligned}
& |\tilde{A}_2(x) - \tilde{A}_2(x')| \\
& \leq \int |\partial^{\nu+\kappa} f(x + l_\alpha(x, v)) - \partial^{\nu+\kappa} f(x' + l_\alpha(x', v))| \prod_{\kappa_i \neq 0, j} |\partial^{\gamma_j^i} l_\alpha^i(x, v)| d\pi \\
& \quad + \sup_x |\partial^{\nu+\kappa} f(x)| \int_{U_1} \left| \prod_{\kappa_i \neq 0, j} \partial^{\gamma_j^i} l_\alpha^i(x, v) - \prod_{\kappa_i \neq 0, j} \partial^{\gamma_j^i} l_\alpha^i(x', v) \right| d\pi \\
& = \tilde{A}_{21} + \tilde{A}_{22}.
\end{aligned}$$

First, by Lemma 14 with  $\lambda = \alpha$ ,

$$\begin{aligned}
\tilde{A}_{21} & \leq \int_{U_1} (|x - x'|^{\beta - [\beta]} + |\Delta l_\alpha(x, x'; v)| \wedge 1) \prod_{\kappa_i \neq 0, j} |\partial^{\gamma_j^i} l_\alpha^i(x, v)| d\pi \\
& \leq C |f|_{\beta+1} \sum_j \int_{U_1} (|x - x'|^{\beta - [\beta]} + |\Delta l_\alpha(x, x'; v)| \wedge 1) [|\partial^{\gamma_j^i} l_\alpha^i|^{\frac{[\beta]}{|\gamma_j^i|} \vee \alpha} + |\partial^{\gamma_j^i} l_\alpha^i|^\alpha] d\pi \\
& \leq C |f|_{\beta+1} |x - x'|^{\beta - [\beta]},
\end{aligned}$$

and

$$\begin{aligned}
\tilde{A}_{22} &\leq |f|_\beta \int_{U_1} \left| \prod_{\kappa_i \neq 0, j} \partial^{\gamma_j^i} l_\alpha^i(x, v) - \prod_{\kappa_i \neq 0, j} \partial^{\gamma_j^i} l_\alpha^i(x', v) \right| d\pi \\
&\leq C|f|_\beta \sum_{\kappa_i \neq 0, j} \left( \int_{U_1} |\Delta \partial^{\gamma_j^i} l_\alpha^i(x, x', v)|^{\frac{[\beta]}{|\gamma_j^i|} \vee \alpha} d\eta \right)^{\left( \frac{|\gamma_j^i|}{[\beta]} \wedge \alpha \right) \theta(\alpha, j)} \\
&\quad \times \left( \int_{U_1} |\Delta \partial^{\gamma_j^i} l_\alpha^i(x, x', v)|^\alpha d\eta \right)^{\alpha(1-\theta(\alpha, j))} \\
&\leq C|f|_\beta |x - x'|^{\beta - [\beta]}.
\end{aligned}$$

*Estimate of  $T_2$ .* The part  $T_2(x)$  consists of the sum whose terms are of the form

$$\int \theta_\alpha(v) \partial^{\nu+\kappa} f(x + l_\alpha) \prod_{\kappa_i \neq 0, j} \partial^{\gamma_j^i} l_\alpha^i d\pi$$

with the non-zero multiindices  $\gamma_j^i \in \mathbf{N}_0^d$  such that  $\sum_{\kappa_i \neq 0, j} \gamma_j^i = \mu$  and  $|\mu| \geq |\kappa| \geq 1$ . By Lemma 14,

$$\begin{aligned}
&\left| \int \theta_\alpha(v) \partial^{\nu+\kappa} f(x + l_\alpha) \prod_{\kappa_i \neq 0, j} \partial^{\gamma_j^i} l_\alpha^i d\pi \right| \\
&\leq \sup_x |\partial^{\nu+\kappa} f(x)| \prod_{\kappa_i \neq 0, j} \left[ \left( \int \theta_\alpha(v) |\partial^{\gamma_j^i} l_\alpha^i| d\pi \right) + \left( \int \theta_\alpha(v) |\partial^{\gamma_j^i} l_\alpha^i|^{\frac{[\beta]}{|\gamma_j^i|}} d\pi \right)^{\frac{|\gamma_j^i|}{[\beta]}} \right] \\
&\leq C|f|_\beta.
\end{aligned}$$

For  $x, x' \in \mathbf{R}^d$ ,

$$\begin{aligned}
&\left| \int \theta_\alpha(v) \partial^{\nu+\kappa} f(x + l_\alpha(x, v)) \prod_{\kappa_i \neq 0, j} \partial^{\gamma_j^i} l_\alpha^i(x, v) d\pi \right. \\
&\quad \left. - \int \theta_\alpha(v) \partial^{\nu+\kappa} f(x' + l_\alpha(x', v)) \prod_{\kappa_i \neq 0, j} \partial^{\gamma_j^i} l_\alpha^i(x', v) d\pi \right| \\
&\leq C \int |\partial^{\nu+\kappa} f(x + l_\alpha(x, v)) - \partial^{\nu+\kappa} f(x' + l_\alpha(x', v))| \prod_{\kappa_i \neq 0, j} |\partial^{\gamma_j^i} l_\alpha^i(x, v)| d\pi \\
&\quad + |f|_\beta \int \left| \prod_{\kappa_i \neq 0, j} \partial^{\gamma_j^i} l_\alpha^i(x, v) - \prod_{\kappa_i \neq 0, j} \partial^{\gamma_j^i} l_\alpha^i(x', v) \right| d\pi \\
&= A + B.
\end{aligned}$$



For the first term, by assumption A3( $\beta$ ) with  $\beta' < \beta$ ,

$$\begin{aligned} A &\leq C|x - x'|^{\beta - [\beta]} |f|_{\beta} \int \theta_{\alpha}(v) \prod_{\kappa_i \neq 0, j} |\partial^{\gamma_j^i} l_{\alpha}^i(x, v)| d\pi \\ &\quad + |f|_{\alpha + \beta'} \int \theta_{\alpha}(v) (|\Delta l_{\alpha}(x, x'; v)| \wedge 1)^{(\alpha + \beta' - [\beta]) \wedge 1} \prod_{\kappa_i \neq 0, j} |\partial^{\gamma_j^i} l_{\alpha}^i(x, v)| d\pi \} \end{aligned}$$

By Lemma 14 with  $\lambda = 1$ ,

$$\begin{aligned} &\int \theta_{\alpha}(v) (|\Delta c(x, x'; v)| \wedge 1)^{(\alpha + \beta' - [\beta]) \wedge 1} \prod_{\kappa_i \neq 0, j} |\partial^{\gamma_j^i} c^i(x, v)| d\pi \\ &\leq \sum_{\kappa_i \neq 0, j} \int \theta_{\alpha}(v) (|\Delta c(x, x'; v)| \wedge 1)^{(\alpha + \beta' - [\beta]) \wedge 1} (|\partial^{\gamma_j^i} c^i| + |\partial^{\gamma_j^i} c^i|^{\frac{[\beta]}{|\gamma_j^i|}}) d\pi \\ &\leq C|x - x'|^{\beta - [\beta]}. \end{aligned}$$

and

$$\int \theta_{\alpha}(v) \prod_{\kappa_i \neq 0, j} |\partial^{\gamma_j^i} l_{\alpha}^i(x, v)| d\pi \leq \sum_{\kappa_i \neq 0, j} \int \theta_{\alpha}(v) (|\partial^{\gamma_j^i} l_{\alpha}^i| + |\partial^{\gamma_j^i} l_{\alpha}^i|^{\frac{[\beta]}{|\gamma_j^i|}}) d\pi \leq C.$$

Hence,  $A \leq C|f|_{\alpha + \beta'} |x - x'|^{\beta - [\beta]}$ .

By lemma 14 with  $\lambda = 1$ ,

$$\begin{aligned} &\int \theta_{\alpha}(v) \left| \prod_{\kappa_i \neq 0, j} \partial^{\gamma_j^i} l_{\alpha}^i(x, v) - \prod_{\kappa_i \neq 0, j} \partial^{\gamma_j^i} l_{\alpha}^i(x', v) \right| d\pi \\ &\leq C \sum_{\kappa_i \neq 0, j} \left\{ \left( \int \theta_{\alpha}(v) |\partial^{\gamma_j^i} l_{\alpha}^i(x, v) - \partial^{\gamma_j^i} l_{\alpha}^i(x', v)|^{\frac{[\beta]}{|\gamma_j^i|}} d\pi \right)^{\frac{|\gamma_j^i|}{[\beta]}} \right. \\ &\quad \left. + \int \theta_{\alpha}(v) |\partial^{\gamma_j^i} l_{\alpha}^i(x, v) - \partial^{\gamma_j^i} l_{\alpha}^i(x', v)| d\pi \right\} \\ &\leq C|x - x'|^{\beta - [\beta]}. \end{aligned}$$

Thus

$$|T_2(x) - T_2(x')| \leq C|f|_{\alpha + \beta'} |x - x'|^{\beta - [\beta]}.$$

The statement follows by the standard interpolation inequalities. ■

### 3.2. Proof of Theorem 9 and Corollary 10

It is well known that for an arbitrary but fixed  $\delta > 0$  there is a family of cubes  $D_k \subseteq \tilde{D}_k \subseteq \mathbf{R}^d$  and a family of  $\eta_k \in C_0^\infty(\mathbf{R}^d)$  with the following properties:

1. For all  $k \geq 1$ ,  $D_k$  and  $\tilde{D}_k$  have a common center  $x_k$ ,  $\text{diam} D_k \leq \delta$ ,  $\text{dist}(D_k, \mathbf{R}^d \setminus \tilde{D}_k) \leq N\delta$  for a certain constant  $N = N(d) > 0$ ,  $\bigcup_k D_k = \mathbf{R}^d$ , and  $1 \leq \sum_k \mathbf{1}_{\tilde{D}_k} \leq 2^d$ .
2. For all  $k$ ,  $0 \leq \eta_k \leq 1$ ,  $\eta_k = 1$  in  $D_k$ ,  $\eta_k = 0$  outside of  $\tilde{D}_k$  and for all multiindices  $\gamma$ ,  $|\partial^\gamma \eta_k| \leq C(d, \delta, |\gamma|) < \infty$ .

For  $\alpha \in (0, 2]$ ,  $\lambda \geq 0$ ,  $k \geq 1$ , denote

$$\begin{aligned} \mathcal{A}^{(\alpha),k} f(x) &= \mathcal{A}_{x_k}^{(\alpha)} f(x), \mathcal{L}_\lambda^{(\alpha),k} f(x) = \left( \frac{\partial}{\partial t} + \mathcal{A}^{(\alpha),k} - \lambda \right) f(x), \\ E^{(\alpha),k} f(x) &= \int [f(x+y) - f(x)] [\eta_k(x+y) - \eta_k(x)] \tilde{m}_\alpha(x_k, y) \frac{dy}{|y|^{d+\alpha}}, \\ E_{k,1}^{(\alpha)} f(x) &= \int [f(x+y) - f(x)] [\eta_k(x+y) - \eta_k(x)] \frac{dy}{|y|^{d+\alpha}}, \\ F^{(\alpha),k} f(x) &= f(x) \mathcal{A}^{(\alpha),k} \eta_k(x), F_1^{(\alpha),k} f(x) = f(x) |\partial|^\alpha \eta_k(x), x \in \mathbf{R}^d. \end{aligned}$$

We will need to estimate these operators.

**Lemma 16.** *Let  $\alpha \in (0, 2]$  and  $\beta > 0$ ,  $\beta \notin \mathbf{N}_0$ . Then*

- a) *for each  $\varepsilon > 0$ , there exists a constant  $C_\varepsilon$  such that for all  $f \in C^\beta(\mathbf{R}^d)$ ,*

$$\sup_k (|E_k^{(\alpha)} f|_\beta + |E_{k,1}^{(\alpha)} f|_\beta) \leq \varepsilon ||\partial|^\alpha f|_\beta + C_\varepsilon |f|_{\beta-[\beta]};$$

- b) *There is a constant  $N = N(\alpha, \beta, d, \delta, M^{(\alpha)})$  such that for all  $f \in C^\beta(\mathbf{R}^d)$ ,*

$$\sup_k (|F_k^{(\alpha)} f|_\beta + |F_{k,1}^{(\alpha)} f|_\beta) \leq N |f|_\beta.$$

**Proof.** For any  $\kappa > 0$

$$\begin{aligned} E^{(\alpha),k} f(x) &= \int_0^1 \int_0^1 \int_{|y| \leq \kappa} (\nabla f(x+sy), y) (\nabla \eta_k(x+ry), y) \mu_k^{(\alpha)}(dy) dr ds \mathbf{1}_{\{\alpha \geq 1\}} \\ &\quad + \int_0^1 \int_{|y| \leq \kappa} [f(x+y) - f(x)] (\nabla \eta_k(x+ry), y) \mu_k^{(\alpha)}(dy) dr \mathbf{1}_{\{\alpha < 1\}} \\ &\quad + \int_{|y| > \kappa} [f(x+y) - f(x)] [\eta_k(x+y) - \eta_k(x)] \mu_k^{(\alpha)}(dy), \end{aligned}$$

where  $\mu_k^{(\alpha)}(dy) = \tilde{m}_\alpha(x_k, y)|y|^{-d-\alpha}dy$ . Clearly,

$$\begin{aligned} |E^{(\alpha),k}f|_\beta &\leq C \left\{ \mathbf{1}_{\{\alpha \geq 1\}} |\nabla f|_{\beta;p} \int_{|y| \leq \kappa} |y|^{-d-\alpha+2} dy \right. \\ &\quad \left. + |f|_\beta \left( \mathbf{1}_{\{\alpha < 1\}} \int_{|y| \leq \kappa} |y|^{-d-\alpha+1} dy + \int_{|y| > \kappa} |y|^{-d-\alpha} dy \right) \right\} \end{aligned}$$

The part b) is straightforward. ■

### 3.2.1. Proof of Theorem 9

It can be easily seen that for any  $f \in C^{\alpha+\beta}(\mathbf{R}^d)$ ,

$$\begin{aligned} \sup_x |f(x)| &\leq \sup_x \sup_k \eta_k(x) |f(x)| = \sup_k \sup_x |\eta_k(x) f(x)|, \\ |f|_\beta &\leq \sup_k |\eta_k f|_\beta + N \sup_x |f(x)|, \sup_k |\eta_k f|_\beta \leq |f|_\beta + N \sup_x |f(x)| \end{aligned} \quad (28)$$

Indeed, for each  $x, y \in \mathbf{R}^d$ ,

$$\begin{aligned} &|\partial^{[\beta]} f(x) - \partial^{[\beta]} f(y)| \\ &= \sup_k \eta_k(x) |\partial^{[\beta]} f(x) - \partial^{[\beta]} f(y)| = \sup_k |\eta_k(x) \partial^{[\beta]} f(x) - \eta_k(x) \partial^{[\beta]} f(y)| \\ &\leq \sup_k |\partial^{[\beta]} \eta_k(x) f(x) - \partial^{[\beta]} \eta_k(y) f(y)| + \sup_k |(\eta_k(y) - \eta_k(x)) u(y)| \\ &\leq \sup_k |\partial^{[\beta]} (\eta_k(x) f(x)) - \partial^{[\beta]} (\eta_k(y) f(y))| \\ &\quad + C |f|_{\beta-1} |x - y|^{\beta-[\beta]}. \end{aligned}$$

The second inequality in (28) then follows. Similarly we can prove the last inequality in (28).

By (28) and Lemma 11 in [15],

$$\begin{aligned} |f|_{\alpha+\beta} &\leq C \sup_x |f(x)| + \|\partial^\alpha f\|_\beta \leq \sup_k |\eta_k| \|\partial^\alpha f\|_\beta + N \sup_x |f(x)| \\ &\leq C [\sup_k \|\partial^\alpha (\eta_k f)\|_\beta + \sup_k |f| \|\partial^\alpha \eta_k + E_1^{(\alpha),k} f\|_\beta]. \end{aligned}$$

and by Lemma 16,

$$|f|_{\alpha+\beta} \leq C \sup_k |\eta_k u|_{\alpha+\beta}. \quad (29)$$

Let  $u \in C^{\alpha+\beta}(H)$  be a solution to (13). Then  $\eta_k u$  satisfies the equation

$$\begin{aligned} \partial_t(\eta_k u) &= \mathcal{A}^{(\alpha),k}(\eta_k u) - \lambda(\eta_k u) + \eta_k [\mathcal{A}^{(\alpha)} u - \mathcal{A}^{(\alpha),k} u] \\ &\quad + \eta_k \mathcal{B}^{(\alpha)} u + \eta_k f + F^{(\alpha),k} u + E^{(\alpha),k} u, \end{aligned} \quad (30)$$

and by Proposition 12,

$$|\eta_k u|_{\alpha+\beta} \leq C[|\eta_k[\mathcal{A}^{(\alpha)}u - \mathcal{A}^{(\alpha),k}u]|_\beta + |\eta_k B^{(\alpha)}u|_\beta + |\eta_k f|_\beta + |F^{(\alpha),k}u|_\beta + |E^{(\alpha),k}u|_\beta].$$

Therefore,

$$|u|_{\alpha+\beta} \leq C[\sup_k |\eta_k f|_\beta + I^{(\alpha)}], \quad (31)$$

where

$$I^{(\alpha)} = |\eta_k[\mathcal{A}^{(\alpha)}u - \mathcal{A}^{(\alpha),k}u]|_\beta + |\eta_k B^{(\alpha)}u|_\beta + |F^{(\alpha),k}u|_\beta + |E^{(\alpha),k}u|_\beta.$$

By Corollary 14 [15],

$$|\eta_k[\mathcal{A}^{(\alpha)}u - \mathcal{A}^{(\alpha),k}u]|_\beta \leq C\delta^\beta |u|_{\alpha+\beta}.$$

Using the estimates of Lemma 16 and Proposition 15, we obtain that for each  $\varepsilon > 0$  there is a constant  $C_\varepsilon$  such that

$$I^{(\alpha)} \leq \varepsilon |u|_{\alpha+\beta} + C_\varepsilon \sup_{t,x} |u|. \quad (32)$$

By (31),

$$|u|_{\alpha+\beta} \leq C[|f|_\beta + |u|_\beta]. \quad (33)$$

On the other hand, (30) holds and by Proposition 12,

$$|u|_\beta \leq \sup_k |\eta_k u|_\beta \leq \mu(\lambda)[|f|_\beta + I_{(\alpha)}],$$

where  $\mu(\lambda) \rightarrow 0$  as  $\lambda \rightarrow \infty$ . Hence, by (32),

$$|u|_\beta \leq C\mu(\lambda)[|f|_\beta + |u|_{\alpha+\beta}]. \quad (34)$$

The inequalities (33) and (34) imply that there exist  $\lambda_0$  with  $0 < \lambda_0 \leq \lambda$  and a constant  $C$  independent of  $u$  such that

$$|u|_{\alpha+\beta} \leq C|f|_\beta \quad (35)$$

If  $u \in C^{\alpha+\beta}(H)$  solves equation (13) with  $\lambda \leq \lambda_0$ , then  $\tilde{u}(t, x) = e^{-(\lambda_0-\lambda)t}u(t, x)$  solves the same equation with  $\lambda_0$ , and by (35),

$$|u|_{\alpha+\beta} \leq e^{(\lambda_0-\lambda)T}|\tilde{u}|_{\alpha+\beta} \leq Ce^{(\lambda_0-\lambda)T}|f|_\beta.$$

Thus (35) holds for all  $\lambda \geq 0$ . Again by Proposition 12 and (29), there is a constant  $C$  such that for all  $s \leq t \leq T$ ,

$$|u(t, \cdot) - u(s, \cdot)|_{\alpha/2+\beta} \leq \sup_k |\eta_k u(t, \cdot) - \eta_k u(s, \cdot)|_{\alpha/2+\beta} \leq C(t-s)^{1/2}[|f|_\beta + |u|_{\alpha+\beta}].$$

Therefore there is a constant  $C$  such that for all  $s \leq t \leq T$ ,

$$|u(t, \cdot) - u(s, \cdot)|_{\alpha/2+\beta} \leq C(t-s)^{1/2}|f|_\beta.$$

Let  $\mathcal{L} = \mathcal{A}_x^{(\alpha)} + \mathcal{B}_x^{(\alpha)} - \lambda, \tau \in [0, 1]$ , and

$$\mathcal{L}_\tau u = \tau \mathcal{L} u + (1 - \tau)|\partial|^\alpha u.$$

We introduce the space  $\tilde{C}^{\alpha+\beta}(H)$  of functions  $u \in C^{\alpha+\beta}(H)$  such that for each  $(t, x)$ ,

$$u(t, x) = \int_0^t F(s, x) ds,$$

where  $F \in C^\beta(H)$ . It is a Banach space with respect to the norm

$$|u|_{\alpha+\beta}^\sim = |u|_{\alpha+\beta} + |F|_\beta.$$

Consider the mappings  $T_\tau : \tilde{C}^{\alpha+\beta}(H) \rightarrow C^\beta$  defined by

$$u(t, x) = \int_0^t F(s, x) ds \mapsto F - \mathcal{L}_\tau u.$$

Obviously, for some constant  $C$  independent of  $\tau$

$$|T_\tau u|_\beta \leq C|u|_{\alpha+\beta}^\sim.$$

On the other hand, there is a constant  $C$  independent of  $\tau$  such that for all  $u \in \tilde{C}^{\alpha+\beta}(H)$

$$|u|_{\alpha+\beta}^\sim \leq C|T_\tau u|_\beta. \quad (36)$$

Indeed,

$$u(t, x) = \int_0^t F(s, x) ds = \int_0^t (L_\tau u + (F - \mathcal{L}_\tau u)) ds.$$

According to (35), there is a constant  $C$  independent of  $\tau$  such that

$$|u|_{\alpha+\beta} \leq C|T_\tau u|_\beta = C|F - \mathcal{L}_\tau u|_\beta. \quad (37)$$

Thus,

$$\begin{aligned} |u|_{\alpha+\beta}^\sim &= |u|_{\alpha+\beta} + |F|_\beta \leq |u|_{\alpha+\beta} + |F - \mathcal{L}_\tau u|_\beta + |\mathcal{L}_\tau u|_\beta \\ &\leq C|u|_{\alpha+\beta} + |F - \mathcal{L}_\tau u|_\beta \leq C|F - \mathcal{L}_\tau u|_\beta = C|T_\tau u|_\beta, \end{aligned}$$

and (36) follows. Since  $T_0$  is an onto map, by Theorem 5.2 in [4], all the  $T_\tau$  are onto maps and the statement follows.

### 3.2.2. Proof of Corollary 10

By Corollary 14 in [15] and Proposition 15, for  $g \in C^{\alpha+\beta}(\mathbf{R}^d)$ ,  $|\mathcal{A}^{(\alpha)}g|_\beta \leq C|g|_{\alpha+\beta}$  and  $|\mathcal{B}^{(\alpha)}g|_\beta \leq C|g|_{\alpha+\beta}$  with a constant  $C$  independent of  $f$  and  $g$ . It then follows from (13) that there exists a unique solution  $\tilde{v} \in C^{\alpha+\beta}(H)$  to the Cauchy problem

$$\begin{aligned} (\partial_t + \mathcal{A}_x^{(\alpha)} + \mathcal{B}_x^{(\alpha)})\tilde{v}(t, x) &= f(t, x) - \mathcal{A}_x^{(\alpha)}g(x) - \mathcal{B}_x^{(\alpha)}g(x), \\ \tilde{v}(T, x) &= 0 \end{aligned} \quad (38)$$

and  $|\tilde{v}|_{\alpha+\beta} \leq C(|g|_{\alpha+\beta} + |f|_\beta)$  with  $C$  independent of  $f$  and  $g$ . Let  $v(t, x) = \tilde{v}(t, x) + g(x)$ , where  $\tilde{v}$  is the solution to problem (38). Then  $v$  is the unique solution to the Cauchy problem (15) and  $|v|_{\alpha+\beta} \leq C(|g|_{\alpha+\beta} + |f|_\beta)$ .

**Remark 17.** *If the assumptions of Corollary 10 hold and  $v \in C^{\alpha+\beta}(H)$  is the solution to (15), then  $\partial_t v = f - \mathcal{A}_x^{(\alpha)}v - \mathcal{B}_x^{(\alpha)}v$ , and according to Corollary 14 in [15] and Proposition 15,  $|\partial_t v|_\beta \leq C(|g|_{\alpha+\beta} + |f|_\beta)$ .*

## 4. One Step Estimate and Proof of the Main Result

The following Lemma provides a one-step estimate of the conditional expectation of an increment of the Euler approximation.

**Lemma 18.** *Let  $\alpha \in (0, 2]$ ,  $\beta > 0$ ,  $\beta \notin \mathbf{N}$ , and  $\delta > 0$ . Assume A1-A4( $\beta$ ) hold. Then there exists a constant  $C$  such that for all  $f \in C^\beta(\mathbf{R}^d)$ ,*

$$|\mathbf{E}[f(Y_s) - f(Y_{\tau_{i_s}})|\mathcal{F}_{\tau_{i_s}}]| \leq C|f|_\beta \delta^{\kappa(\alpha, \beta)}, \forall s \in [0, T],$$

where  $i_s = i$  if  $\tau_i \leq s < \tau_{i+1}$  and  $\kappa(\alpha, \beta)$  is as defined in Theorem 1.

The proof of Lemma 18 is based on applying Itô's formula to  $f(Y_s) - f(Y_{\tau_{i_s}})$ ,  $f \in C^\beta(\mathbf{R}^d)$ . If  $\beta > \alpha$ , by Remark 6 and Itô's formula, the inequality holds. If  $\beta < \alpha$ , we first smooth  $f$  by using  $w \in C_0^\infty(\mathbf{R}^d)$ , a nonnegative smooth function with support on  $\{|x| \leq 1\}$  such that  $w(x) = w(|x|)$ ,  $x \in \mathbf{R}^d$ , and  $\int w(x)dx = 1$  (see (8.1) in [3]). Note that, because of the symmetry,

$$\int_{\mathbf{R}^d} x^i w(x)dx = 0, i = 1, \dots, d. \quad (39)$$

For  $x \in \mathbf{R}^d$  and  $\varepsilon \in (0, 1)$ , define  $w^\varepsilon(x) = \varepsilon^{-d}w(\frac{x}{\varepsilon})$  and the convolution

$$f^\varepsilon(x) = \int f(y)w^\varepsilon(x - y)dy = \int f(x - y)w^\varepsilon(y)dy, x \in \mathbf{R}^d. \quad (40)$$

#### 4.1. Some Auxiliary Estimates

In [15] the following estimates for  $\mathcal{A}_z^{(\alpha)}$  and  $f^\varepsilon$  were proved.

**Lemma 19.** (Lemma 21 in [15]) *Let  $\alpha \in (0, 2)$ ,  $\beta < \alpha$ ,  $\beta \neq 1$ , and  $\varepsilon \in (0, 1)$ . Then*

(i) *there exists a constant  $C$  such that for all  $f \in C^\beta(\mathbf{R}^d)$ ,  $x \in \mathbf{R}^d$ ,*

$$|f^\varepsilon(x) - f(x)| \leq C\varepsilon^\beta |f|_\beta;$$

(ii) *there exists a constant  $C$  such that for all  $z, x \in \mathbf{R}^d$ ,*

$$|\mathcal{A}_z^{(\alpha)} f^\varepsilon(x)| \leq C\varepsilon^{-\alpha+\beta} |f|_\beta \quad (41)$$

*and in particular, for all  $f \in C^\beta(\mathbf{R}^d)$ ,  $z, x \in \mathbf{R}^d$ ,*

$$|\partial^\alpha f^\varepsilon(x)| \leq C\varepsilon^{-\alpha+\beta} |f|_\beta; \quad (42)$$

(iii) *for  $k, l = 1, \dots, d$ ,  $x \in \mathbf{R}^d$ ,*

$$\begin{aligned} |\partial_k f^\varepsilon(x)| &\leq C\varepsilon^{-1+\beta} |f|_\beta, \text{ if } \beta < 1, \\ |f^\varepsilon|_1 &\leq C|f|_1, \\ |\partial_{kl}^2 f^\varepsilon(x)| &\leq C\varepsilon^{-2+\beta} |f|_\beta, \text{ if } \beta < 2, \end{aligned} \quad (43)$$

*and*

$$|f^\varepsilon|_\alpha \leq C\varepsilon^{-\alpha+\beta} |f|_\beta, \text{ if } \alpha \in (1, 2), \beta \in (0, 1], \quad (44)$$

$$|\partial^{\alpha-1} \nabla f^\varepsilon(x)| \leq C\varepsilon^{-\alpha+\beta} |f|_\beta, \text{ if } \alpha \in (1, 2), \beta \in (1, \alpha). \quad (45)$$

**Corollary 20.** *Assume  $a(x)$  and*

$$\int [\mathbf{1}_{U_1}(v) |l_\alpha(x, v)|^\alpha + \mathbf{1}_{U_1^c}(v) |l_\alpha(x, v)|^{\alpha \wedge 1} \wedge 1] \pi(dv),$$

*are bounded,  $\varepsilon \in (0, 1)$ . Then there exists a constant  $C$  such that for all  $z, x \in \mathbf{R}^d$ ,  $f \in C^\beta(\mathbf{R}^d)$ ,*

$$|\mathcal{B}_z^{(\alpha)} f^\varepsilon(x)| \leq C\varepsilon^{-\alpha+\beta} |f|_\beta.$$

**Proof.** If  $\beta < \alpha < 1$ , by Lemma 13,

$$f^\varepsilon(x+y) - f^\varepsilon(x) = \int k^{(\alpha)}(y, y') \partial^\alpha f^\varepsilon(x-y') dy',$$

and by Lemma 19, (42),

$$|f^\varepsilon(x+y) - f^\varepsilon(x)| \leq C\varepsilon^{-\alpha+\beta}|f|_\beta(|y|^\alpha \wedge 1), x, y \in \mathbf{R}^d \quad (46)$$

and

$$\begin{aligned} |f^\varepsilon(x+l_\alpha(x,v)) - f^\varepsilon(x)| &\leq C\varepsilon^{-\alpha+\beta}|f|_\beta(|l_\alpha(x,v)|^\alpha \wedge 1) \\ &\leq C\varepsilon^{-\alpha+\beta}|f|_\beta[\mathbf{1}_{U_1}(v)|l_\alpha(x,v)|^\alpha + \mathbf{1}_{U_1^c}(v)(|l_\alpha(x,v)|^\alpha \wedge 1)]. \end{aligned}$$

If  $\beta < \alpha = 1$ , by Lemma 19(ii) and (43),

$$\begin{aligned} |f^\varepsilon(x+y) - f^\varepsilon(x)| &\leq C \sup_x [f(x) + |\nabla f^\varepsilon(x)|] (|y| \wedge 1) \quad (47) \\ &\leq C\varepsilon^{-1+\beta}|f|_\beta(|y| \wedge 1), x, y \in \mathbf{R}^d \end{aligned}$$

and

$$\begin{aligned} |f^\varepsilon(x+l_1(x,v)) - f^\varepsilon(x)| &\leq C\varepsilon^{-1+\beta}|f|_\beta(|l_1(x,v)| \wedge 1) \\ &\leq C\varepsilon^{-1+\beta}|f|_\beta[\mathbf{1}_{U_1}(v)|l_1(x,v)| + \mathbf{1}_{U_1^c}(v)(|l_1(x,v)| \wedge 1)]. \end{aligned}$$

Assume  $\alpha \in (1, 2)$ , then for  $x, y \in \mathbf{R}^d$ ,

$$f^\varepsilon(x+y) - f^\varepsilon(x) - (\nabla f^\varepsilon(x), y) = \int_0^1 (\nabla f^\varepsilon(x+sy) - \nabla f^\varepsilon(x), y) ds. \quad (48)$$

If  $\beta \in (1, \alpha)$ , then by Lemmas 13, 19 and (44), for  $x, y' \in \mathbf{R}^d$ ,

$$\begin{aligned} |\nabla f^\varepsilon(x+y') - \nabla f^\varepsilon(x)| &\leq C \sup_x |\partial^{\alpha-1} \nabla f^\varepsilon(x)| |y'|^{\alpha-1} \quad (49) \\ &\leq C\varepsilon^{-\alpha+\beta}|f|_\beta|y'|^{\alpha-1}. \end{aligned}$$

If  $\beta > \alpha > 1$ , then directly

$$|\nabla f^\varepsilon(x+y') - \nabla f^\varepsilon(x)| \leq C|f|_\beta|y'|^{\alpha-1}.$$

If  $\beta \in (0, 1]$ ,  $\alpha \in (1, 2)$ , then by Lemma 19, (45),

$$|\nabla f^\varepsilon(x+y') - \nabla f^\varepsilon(x)| \leq C\varepsilon^{-\alpha+\beta}|y'|^{\alpha-1}|f|_\beta \quad (50)$$

Applying (49), (50) to (48) we have for  $x, y \in \mathbf{R}^d$ ,

$$|f^\varepsilon(x+y) - f^\varepsilon(x) - (\nabla f^\varepsilon(x), y)| \leq C\varepsilon^{-\alpha+\beta}|y|^\alpha|f|_\beta.$$

Hence,

$$\mathbf{1}_{U_1}(v)|f^\varepsilon(x+l_\alpha(x,v)) - f^\varepsilon(x) - (\nabla f^\varepsilon(x), l_\alpha(x,v))| \leq C\varepsilon^{-\alpha+\beta}|l_\alpha(x,v)|^\alpha|f|_\beta.$$

Also, for  $\alpha > 1, \beta \in (1, \alpha)$ ,

$$\mathbf{1}_{U_1^c}(v)|f^\varepsilon(x+l_\alpha(x,v)) - f^\varepsilon(x)| \leq C|f|_\beta(|l_\alpha(x,v)| \wedge 1).$$

Therefore, the statement follows by the assumptions and Lemma 19. ■



#### 4.2. Proof of Lemma 18

If  $\beta < \alpha$ , define  $f^\varepsilon$  by (40) for  $\varepsilon \in (0, 1)$  and apply Itô's formula (see Remark 6): for  $s \in [0, T]$ ,

$$\mathbf{E}[f^\varepsilon(Y_s) - f^\varepsilon(Y_{\tau_{is}})|\mathcal{F}_{\tau_{is}}] = \mathbf{E}\left[\int_{\tau_{is}}^s (\mathcal{A}_{Y_{\tau_{is}}}^{(\alpha)} f^\varepsilon(Y_r) + \mathcal{B}_{Y_{\tau_{is}}}^{(\alpha)} f^\varepsilon(Y_r))dr|\mathcal{F}_{\tau_{is}}\right].$$

Hence, by Lemma 19 and Corollary 20, for  $\varepsilon \in (0, 1)$ ,

$$\begin{aligned} |\mathbf{E}[f(Y_s) - f(Y_{\tau_{is}})|\mathcal{F}_{\tau_{is}}]| &\leq |\mathbf{E}[(f - f^\varepsilon)(Y_s) - (f - f^\varepsilon)(Y_{\tau_{is}})|\mathcal{F}_{\tau_{is}}]| \\ &\quad + |\mathbf{E}[f^\varepsilon(Y_s) - f^\varepsilon(Y_{\tau_{is}})|\mathcal{F}_{\tau_{is}}]| \\ &\leq C(\varepsilon^\beta + \delta\varepsilon^{-\alpha+\beta})|f|_\beta, \end{aligned}$$

with a constant  $C$  independent of  $\varepsilon, f$ . Minimizing  $\varepsilon^\beta + \delta\varepsilon^{-\alpha+\beta}$  in  $\varepsilon \in (0, 1)$ , we obtain

$$|\mathbf{E}[f(Y_s) - f(Y_{\tau_{is}})|\mathcal{F}_{\tau_{is}}]| \leq C\delta^{\kappa(\alpha, \beta)}|f|_\beta.$$

If  $\beta > \alpha$ , we apply Itô's formula directly (see Remark 6):

$$\mathbf{E}[f(Y_s) - f(Y_{\tau_{is}})|\mathcal{F}_{\tau_{is}}] = \mathbf{E}\left[\int_{\tau_{is}}^s (\mathcal{A}_{Y_{\tau_{is}}}^{(\alpha)} f(Y_r) + \mathcal{B}_{Y_{\tau_{is}}}^{(\alpha)} f(Y_r))dr|\mathcal{F}_{\tau_{is}}\right].$$

Hence, by Corollary 14 in [15] and Lemma 19,

$$|\mathbf{E}[f(Y_s) - f(Y_{\tau_{is}})|\mathcal{F}_{\tau_{is}}]| \leq C\delta|f|_\beta.$$

The statement of Lemma 18 follows.

#### 4.3. Proof of Theorem 1

Let  $v \in C^{\alpha+\beta}(H)$  be the unique solution to (15) (see Corollary 10). By Itô's formula (see Remark 6, (11)) and (15)),

$$\begin{aligned} \mathbf{E}v(0, X_0) &= \mathbf{E}v(T, X_T) - \mathbf{E}\int_0^T (\partial_t v(s, X_s) + \mathcal{A}_{X_s}^{(\alpha)} v(s, X_s) + \mathcal{B}_{X_s}^{(\alpha)} v(s, X_s))ds \\ &= \mathbf{E}[g(X_T) - \int_0^T f(X_s)ds], \end{aligned}$$

and

$$\mathbf{E}v(0, X_0) = \mathbf{E}v(0, Y_0). \quad (51)$$

By Proposition 15, Corollaries 14 in [15], 10 and Remark 17,

$$\begin{aligned} |\mathcal{A}_z^{(\alpha)} v(s, \cdot)|_\beta + |\mathcal{B}_z^{(\alpha)} v(s, \cdot)|_\beta &\leq C|v|_{\alpha+\beta} \leq C|g|_{\alpha+\beta}, \\ |\partial_t v(s, \cdot)|_\beta &\leq C|g|_{\alpha+\beta}, s \in [0, T]. \end{aligned} \quad (52)$$

Then, by Itô's formula (Remark 6, (11)) and Corollary 10, with (51) and (52), it follows that

$$\begin{aligned}
& \mathbf{E}g(Y_T) - \mathbf{E}g(X_T) - \mathbf{E} \int_0^T f(Y_{\tau_{i_s}})ds + \mathbf{E} \int_0^T f(X_s)ds \\
&= \mathbf{E}v(T, Y_T) - \mathbf{E}v(0, Y_0) - \mathbf{E} \int_0^T f(Y_{\tau_{i_s}})ds + \mathbf{E} \int_0^T f(X_s)ds \\
&= \mathbf{E} \int_0^T \{ [\partial_t v(s, Y_s) - \partial_t v(s, Y_{\tau_{i_s}})] \\
&\quad + [\mathcal{A}_{Y_{\tau_{i_s}}}^{(\alpha)} v(s, Y_s) - \mathcal{A}_{Y_{\tau_{i_s}}}^{(\alpha)} v(s, Y_{\tau_{i_s}})] \\
&\quad + [\mathcal{B}_{Y_{\tau_{i_s}}}^{(\alpha)} v(s, Y_s) - \mathcal{B}_{Y_{\tau_{i_s}}}^{(\alpha)} v(s, Y_{\tau_{i_s}})] \} ds.
\end{aligned}$$

Hence, by (52) and Lemma 18, there exists a constant  $C$  independent of  $g$  such that

$$|\mathbf{E}g(Y_T) - \mathbf{E}g(X_T)| \leq C\delta^{\kappa(\alpha, \beta)}|g|_{\alpha+\beta}.$$

The statement of Theorem 1 follows.

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